THE STRAIGHT LINE

Definition of a Straight line

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

Theorem : Prove that every first degree equation in x, y represents a straight line.

Proof : Let $ax + by + c = 0$ be a first degree equation in x, y where a, b, c are constants. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$. Then

$$ax_1 + by_1 + c = 0 \text{ and } ax_2 + by_2 + c = 0 \quad \ldots(1)$$

Let R be any point on the line segment joining P and Q. suppose R divides PQ in the ratio $\lambda : 1$ Then the coordinates of R are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

We have $a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c$

$$\Rightarrow \quad \frac{\lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c)}{\lambda + 1} = \lambda\, 0 + 0 = 0$$

$\therefore$ $R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by $ax + by + c = 0$

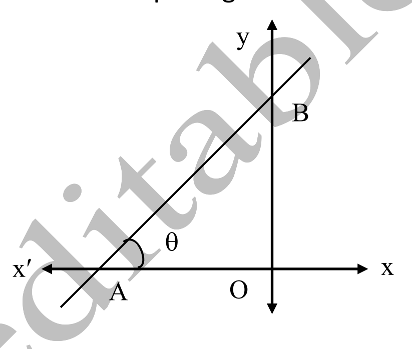
Thus, every point on the line segment joining P and Q lies on $ax + by + c = 0$.

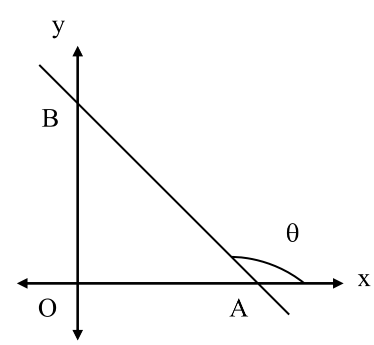
Hence, $ax + by + c = 0$ represents a straight line.

Note : When we say that a first degree equation in x, y, i.e., $ax + by + c = 0$ represents a line, it means that all points (x, y) satisfying $ax + by + c = 0$ lie along a line. Thus, a line is also defined as the locus of a point satisfying the condition $ax + by + c = 0$ where a, b, c are constants.

Slope(Gradient) of a line

The trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called the slope or gradient of the line.





The slope of a line is generally denoted by m. Thus, $m = \tan\theta$

(i) Since a line parallel to x-axis makes an angle of 0° with x-axis, therefore its slope is $\tan 0^\circ = 0$.

(ii) A line parallel to y-axis i.e., perpendicular to x-axis makes an angle of 90° with x-axis, so its slope is not defined.

(iii) Also the slope of a line equally inclined with axes is 1 or –1 as it makes 45° or 135° angle with positive direction of x-axis.

Remark: The angle of inclination of a line with the positive direction of x-axis in anticlockwise sense always lies between 0° and 180°.

Illustration 1

What can be said regarding a line if its slope is

(i) positive (ii) zero (iii) negative ?

Let θ be the angle of inclination of the given line with the positive direction in anticlockwise sense. Then, its slope is given by $m = \tan\theta$

(i) Slope is positive $\Rightarrow m = \tan\theta > 0$

$\Rightarrow$ θ lies between 0° and 90°

$\Rightarrow$ θ is an acute angle

(ii) Slope is zero $\Rightarrow m = \tan\theta = 0$

$\Rightarrow$ $\theta = 0^\circ$

$\Rightarrow$ either the line is x-axis or it is parallel to x-axis

(iii) Slope is negative $\Rightarrow m = \tan\theta < 0$

$\Rightarrow$ θ lies between 90° and 180°

$\Rightarrow$ θ is an obtuse angle.

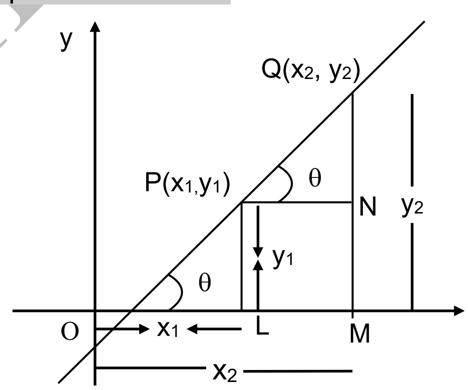
Slope of a line in terms of coordinates of any two points on it

Let P (x_1, y_1) and Q (x_2, y_2) be two points on a line making an angle θ with the positive direction of x-axis. Draw PL, QM perpendiculars on x-axis and PN $\perp$ QM. Then

PN = LM = OM – OL = $x_2 - x_1$

and QN = QM – NM = QM – PL = $y_2 - y_1$.

In Δ PQN, $\tan\theta = \dfrac{QN}{PN} = \dfrac{y_2 - y_1}{x_2 - x_1}$



Thus, if (x_1, y_1) and (x_2, y_2) are coordinates of any two points on a line, then its slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}.$$

Note :

(i) If $x_1 = x_2$ then m is not defined. In this case line is perpendicular to x-axis.

(ii) If three points A, B, C are collinear, then slope of AB = slope of BC = slope of AC.

Illustration 2

Find the slope of a line which passes through points (3,2) and (–1,5).

We know that the slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here the line passes through (3, 2) and (–1, 5). So, its slope is given by $m = \dfrac{5-2}{-1-3} = -\dfrac{3}{4}$.

Angle between two lines

The angle θ between the lines having slopes m_1 and m_2 is given by

$\tan\theta = \pm\dfrac{m_2 - m_1}{1 + m_1 m_2}$

PROOF: Let m_1 and m_2 be the slope of two given lines AB and CD which intersect at a point P and make angles θ_1 and θ_2 respectively with the positive direction of x-axis. Then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

Let $\angle APC = \theta$ be the angle between the given lines. Then

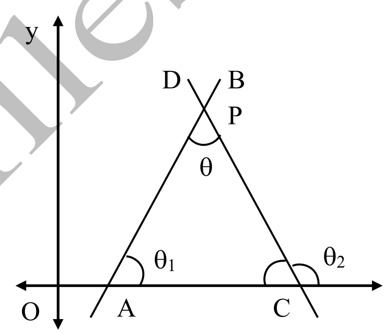
$\theta_2 = \theta + \theta_1$

$\Rightarrow \quad \theta = \theta_2 - \theta_1$

$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$

$\Rightarrow \quad \tan\theta = \dfrac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$ .....(i)

$\Rightarrow \quad \tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$ $\left[\text{Using } \tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}\right]$



Since, $\angle APD = \pi - \theta$ is also the angle between AB and CD, therefore

$\tan\angle APD = \tan(\pi - \theta)$

$= -\tan\theta = -\dfrac{m_2 - m_1}{1 + m_1 m_2}$ ....(ii)

From (i) and (ii), we find that the angle between two lines of slopes m_1 and m_2 is given by

$\tan\theta = \pm\left(\dfrac{m_2 - m_1}{1 + m_1 m_2}\right) \Rightarrow \theta = \tan^{-1}\left(\pm\dfrac{m_2 - m_1}{1 + m_1 m_2}\right)$

The acute angle between the lines is given by

$\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$.

Illustration 3

If A (–2, 1), B (2, 3) and C (–2, – 4) are three points, find the angle between BA & BC.

Let m_1 and m_2 be the slopes of BA and BC respectively. Then

$m_1 = \dfrac{3-1}{2-(-2)} = \dfrac{2}{4} = \dfrac{1}{2}$

and $m_2 = \dfrac{-4-3}{-2-2} = \dfrac{7}{4}$

Let θ be the angle between BA and BC. Then

$\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\dfrac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right| = \left|\dfrac{\frac{10}{8}}{\frac{15}{8}}\right| = \pm\dfrac{2}{3}$

Condition for parallelism of lines

It two lines of slopes m_1 and m_2 are parallel, then angle θ between them is of 0°.

$\therefore \quad \tan\theta = \tan 0^\circ = 0$

$\Rightarrow \quad \dfrac{m_2 - m_1}{1 + m_1 m_2} = 0$ $\left[\text{Using } \tan\theta = \pm\dfrac{m_2 - m_1}{1 + m_1 m_2}\right]$

$\Rightarrow \quad m_2 = m_1$

Thus when two lines are parallel, their slopes are equal.

Condition for perpendicularity of two lines

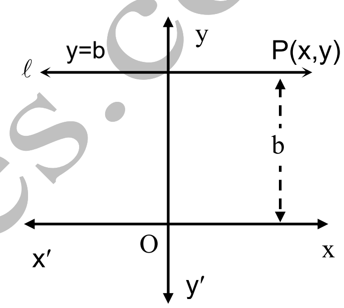
If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is of 90º

$$\therefore \quad \cot\theta \Rightarrow \frac{1+m_1m_2}{m_2-m_1} = 0 \Rightarrow m_1\,m_2 = -1$$

Thus when two lines are perpendicular, the product of their slopes is –1. If m is the slope of a line, then the slope of a line perpendicular to it is – (1/m).

Equation of lines parallel to the coordinate axis

Equation of a line parallel to x-axis



Let ℓ be a straight line parallel to x-axis at a distance b from it. Then, clearly the ordinate of each point on ℓ is b. thus, ℓ can be considered as the locus of a point at a distance b from x-axis. Then, if P (x, y) is any point on ℓ, then y = b.

The equation of a line parallel to x-axis at a distance b from is y = b.

Since x-axis is parallel to itself at a distance 0. From it, the equation of x-axis is y = 0.

If a line is parallel to x-axis at a distance b and below x-axis, then its equation is y = – b

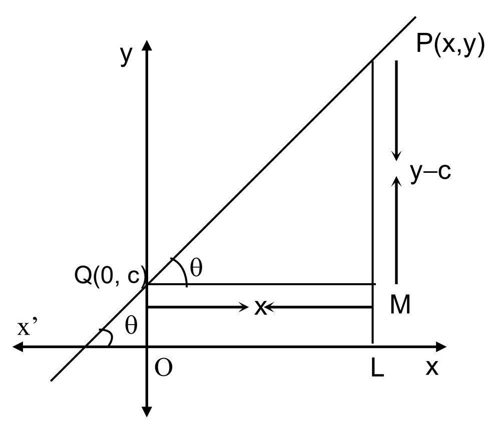
Equation of a line parallel to y-axis

As above, equation of a line parallel to y-axis is x = a where a is the distance of the line from y-axis.

Different forms of the equation of a straight line

The slope intercept form of a line

The equation of a line with slope m and making an intercept c on y-axis is y = mx + c.



PROOF: let the given line intersect y-axis at Q and makes an angle θ with x-axis. Then $m = \tan\theta$. Let P (x, y) be any point on the line. Draw PL perpendicular to x-axis and QM $\perp$ PL.

Clearly $\angle MQP = \theta$, QM = OL = x

and PM = PL – ML = PL – OQ = y – c.

From triangle PMQ, we have

$$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x} \Rightarrow m = \frac{y-c}{x}$$

$\Rightarrow y = mx + c$

Which is the required equation of the line .

REMARK : 1. If the line passes through the origin, then $0 = m0 + c \Rightarrow c = 0$. Therefore the equation of a line passing through the origin is y = mx, where m is the slope of the line.

REMARK : 2. If the line parallel to x-axis, then m = 0, there fore the equation of a line parallel to x-axis is y = c

The point-slope form of a line

The equation of a line, which passes through the point (x_1, y_1) and has the slope 'm' is

$$y - y_1 = m(x - x_1)$$

PROOF: Let Q (x_1, y_1) be the point through which the line passes and let P (x, y) be any point on the line. Then slope of the line is

$$\frac{y - y_1}{x - x_1}$$

But m is the slope of the line. There

$$m = \frac{y - y_1}{x - x_1} \Rightarrow y - y_1 = m\,(x - x_1)$$

Thus, $y - y_1 = m\,(x - x_1)$ is the required equation of the line.

The two-point form of a line

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$.

PROOF: Let m be the slope of the line passing through (x_1, y_1) and (x_2, y_2). Then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m\,(x - x_1) \qquad \text{[using point-slope form]}$$

Substituting the value of m, we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

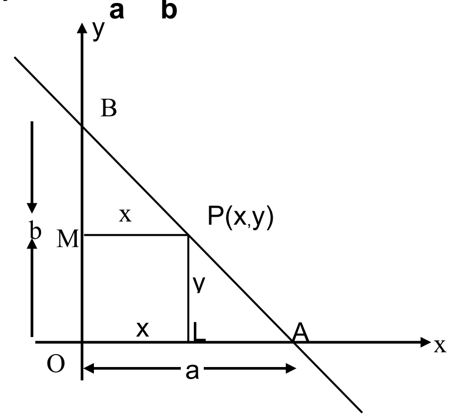
This is the required equation of the line in two point from.

Note : This equation fails to apply when x-coordinates of points are same. In this case equation is $x = x_1$ or $x = x_2$

The Intercept form of a line

The equation of a line which cuts off intercepts a and b respectively from the x and y-axes is $\frac{x}{a} + \frac{y}{b} = 1$.

PROOF: Let AB be the line which cuts off intercepts OA = a and OB = b on the x and y axes respectively. Let (x, y) be any point on the line. Draw PL $\perp$ OX. Then OL = x and PL = y.



Clearly,

Area of Δ OAB = Area of Δ OPA + Area of Δ OPB

$$\Rightarrow \frac{1}{2}\text{OA. OB} = \frac{1}{2}\text{OA. PL} + \frac{1}{2}\text{OB. PM}$$

$$\Rightarrow \frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx$$

$$\Rightarrow ab = ay + bx \Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$$

Illustration 4

Find the equation of the line which passes through the point (3,4) and the sum of its intercepts on the axes is 14.

Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ ...(1)

This passes through (3,4) therefore

$$\frac{3}{a} + \frac{4}{b} = 1 \qquad ...(2)$$

It is given that a + b = 14 $\Rightarrow$ b = 14 – a. putting b = 14 – a in (2) we get,

$$\frac{3}{a} + \frac{4}{14-a} = 1 \quad \Rightarrow a^2 - 13a + 42 = 0 \quad \Rightarrow \quad (a-7)(a-6) = \Rightarrow a = 7, 6$$

for a = 7, b = 14 – 7=7 and for a = 6, b = 14 – 6 = 8.

Putting the values of a and b in (1), we get the equations of the lines

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{x}{6} + \frac{y}{8} = 1 \quad \text{or} \quad x + y = 7 \text{ and } 4x + 3y = 24.$$

Illustration 5

Find the equation of the line through (2, 3) so that the segment of the line intercepted between the axes is bisected at this point.

Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ which meets the x and y axes at A (a, 0) and B (0, b) respectively. The coordinates of the mid-point of AB are $\left(\frac{a}{2}, \frac{b}{2}\right)$

$$\frac{a}{2} = 2 \text{ and } \frac{b}{2} = 3 \Rightarrow a = 4 \text{ and } b = 6.$$

Hence, the equation of the required line is $\frac{x}{4} + \frac{y}{6} = 1$ or $3x + 2y = 12$.

Illustration 6

Find the equation of the line which cuts off equal and positive intercepts from the axes and passes through the point (α, β).

Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ which cuts off intercepts **a** and **b** with the coordinate axes.

It is given that a = b. therefore the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ or, } x + y = a \qquad ...(1)$$

But it passes through (α, β). So, $\alpha + \beta = a$.

Putting the value of **a** in (1), the equation of the line is $x + y = \alpha + \beta$

PRACTICE ASSIGNMENT I

1. Find the slope of the line joining the following pairs of points :

(i) (3, 5) and (5, 9) (ii) (4, –7) and (1, 2)

(iii) (4, –6) and (–2, –5) (iv) (–5, –3) and (3, 1)

2.
(i) Find the angle between the line joining the points (0, 0) and (2, 3) and the points (2, –2) and (3, 5).

(ii) Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.

(iii) Find the angle between the x-axis and the line joining the points (3, –1) and (4, –2).

(iv) The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.

(v) A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that $k - y_1 = m(h - x_1)$.

3. Find the equation of the line:

(i) parallel to the x-axis and passing through (2, 5);

(ii) parallel to the y-axis and passing through (–3, 2).

4. Write down the equation of the line

(i) which passes through (2, 1) having a slope of $\frac{1}{2}$;

(ii) whose slope is 2.5 and which makes an intercept of 3.5 units on the y-axis.

5. Find the equation of the line passing through the pair of points:

(i) (0, 0) and (3, –2); (ii) (2, 0) and (0, –3) ;

(iii) (–2, 5) and (1, 2) ; (iv) (–7, –8) and (–4, –1) ;

(v) (–1, –2) and (–5, 2);

6. (i) Find the equation of the line joining the points (2, 4) and (7, 9) and hence prove that the three points (2, 4), (7, 9) and (–1, 1) are collinear.

(ii) If three points (h, 0), (a,b) and (0, k) lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

7. Find the equation of a line which passes through the point (–3, 2) and makes angle of 30° with the positive direction of the x-axis.

8. (i) Find the equation of the straight line through (2, 5) and making equal intercepts of opposite signs on the axes.

(ii) Find the equation of the straight line making intercepts on the axes equal in magnitude but opposite in sign and passing through the point (–3, –5).

9. A line passes through the point (12, –1) and the sum of the intercepts made by the line on the axes is 7. Find the equation of the line.

10. A straight line is drawn through the point (3, 5) such that the point bisects the portion of the line intercepted between the axes. Find the equation of the line.

11. A straight line has the equation $x - 2y + 4 = 0$. What is its gradient ? y-intercepts? x-intercept? What angle does the line make with $x + 3y = 0$?

12. Find the equation of the line joining the points (3, –1) and (2, 3). Also find the equation of another line perpendicular to this line and passing through the point (5, 2).

13. Find the equation of the line perpendicular to the line $3x + 2y = 8$ and passing through the mid-point of the segment joining (2, 7) and (–4, 1).

14. A is the point (2, 0) on the x-axis and B is the point (6, 2). The perpendicular to AB at A meets the y-axis at C. Find the gradient of AC and the co-ordinates of the middle point of AC.

15. Prove that the points whose co-ordinates are respectively (5, 1) (1, –1) and (11, 4) lie on a straight line and find the intercepts of this line on the axes.

16. Find the equation of the straight line passing through the point (4, 9) which divides the portion of the line intercepted between the axes in the ratio 2 : 3.

17. The vertices of a Δ ABC are A (0, 0), B(1, 5) and C(–2, 2). Find the equation of the altitude through A.

18. Find the equations of the diagonals of the rectangle whose sides are $x = 2$, $x = -1$, $y = 6$ and $y = -2$.

19. As the number of units manufactured increases from 5,000 to 7,000, the total cost of production increases from Rs. 26,000 to Rs. 34,000. Find the relationship between the cost (y) and the number of units made (x), if the relationship is linear.

20. A firm produces 200 units of a product for a total cost of Rs. 730 and 500 units of the product for a total cost of Rs. 970. Assuming the cost curve to be linear, derive the equation of this line and use it to estimate the cost of producing 400 units of the product.

21. The vertices of Δ PQR are P (2, 1), Q(–2, 3) and R (4, 5). Find equation of the median through the vertex R.

22. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the point (2, 5) and (–3, 6).

23. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: n. Find the equation of the line.

24. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3)

25. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.

26. Reduce the following equations into slope-intercept from and find their slopes and the y-intercepts.
(i) $x + 7y = 0$ (ii) $6x + 3y - 5 = 0$ (iii) $y = 0$

27. Reduce the following equations into intercept from and find their intercepts on the axes.
(i) $3x + 2y - 12 = 0$ (ii) $4x - 3y = 6$ (iii) $3y + 2 = 0$

28. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$

29. Two lines passing through the point (2, 3) intersects each other at an angle of 60°. If slope of one line is 2, find equation of the other line.

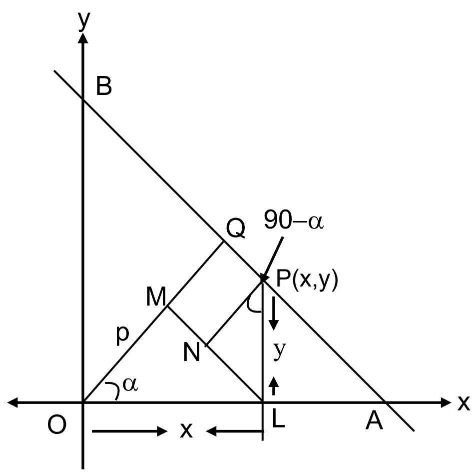
30. A ray of light passing through the point (1, 2) reflects on the x axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

The normal form or perpendicular form of a line

The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with x-axis is $x \cos \alpha + y \sin \alpha = p$

PROOF: Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be p and $\angle LOQ = \alpha$.

Let P (x, y) be any point on the line. Draw PL $\perp$ OA, LM $\perp$ OQ and PN $\perp$ LM. Then OL = x and LP = y.



In ΔOLM, $\cos \alpha = \dfrac{OM}{OL}$

$\Rightarrow$ $OM = OL \cos \alpha = x \cos \alpha$

In ΔPNL, $\cos(90 - \alpha) = \dfrac{PN}{PL} \Rightarrow \sin \alpha = \dfrac{PN}{PL}$

$\Rightarrow$ $PN = PL \sin \alpha = y \sin \alpha$

$\Rightarrow$ $MQ = PN = y \sin \alpha$

Now, $p = OQ = OM + MQ = x \cos \alpha + y \sin \alpha$

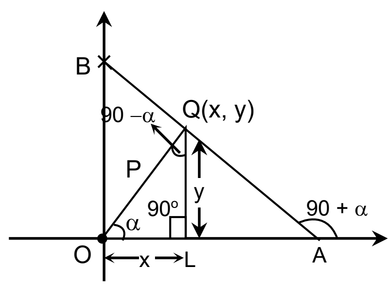
So, the equation of the required line is $x \cos \alpha + y \sin \alpha = p$.

ALITER

Let the line AB be straight line the length of the perpendicular OQ from the origin to AB is p and $\angle AOQ = \alpha$

Now Draw QL $\perp$ OA

$\dfrac{x}{p} = \cos\alpha \Rightarrow x = p \cos \alpha$, $\dfrac{y}{p} = \sin\alpha$ $\Rightarrow y = p \sin \alpha$



$\therefore$ Q (p cos α, p sin α)

$\therefore$ Equation of line AB is

$y - p \sin \alpha = \tan (90 + \alpha) (x - p \cos \alpha)$

$\Rightarrow$ $y - p \sin \alpha = -\cot\alpha (x - p \cos\alpha)$

$\Rightarrow \quad y - p\sin\alpha = \dfrac{-\cos\alpha}{\sin\alpha}(x - p\cos\alpha)$

$\Rightarrow \quad \sin\alpha\, y - p\sin^2\alpha = -(\cos\alpha)x + p\cos^2\alpha$

$\Rightarrow \quad (\cos\alpha)x + (\sin\alpha)y = p(\cos^2\alpha + \sin^2\alpha)$

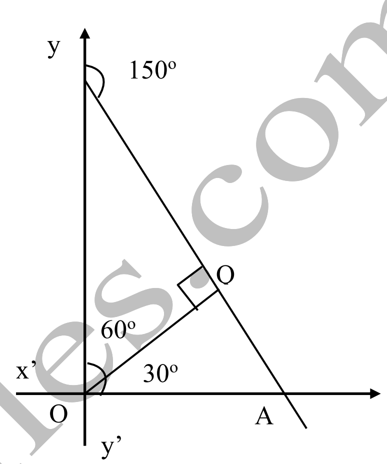
$\Rightarrow \quad (\cos\alpha)x + (\sin\alpha)y = p$

Note : (i) Distance p is always taken as positive.

(ii) (Coefficient of x)^2 + (Coefficient of y)^2 = 1

Illustration 7

The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of y-axis. Find the equation of the line.



Here p = 7 and $\alpha = 30°$

$\therefore$ equation of the required line is

$$x\cos 30° + y\sin 30° = 7$$

or, $\quad x\dfrac{\sqrt{3}}{2} + y\cdot\dfrac{1}{2} = 7$

or, $\quad \sqrt{3}\,x + y = 14.$

The distance form of a line or parametric form of a line

The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is

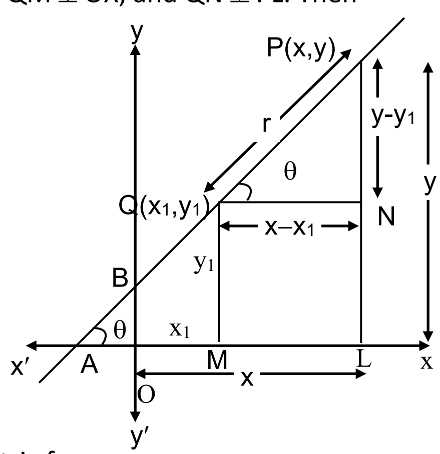
$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

where r is the distance of the point (x, y) on the line from the point (x_1, y_1)

PROOF: Let the give line meet x-axis at A, y-axis at B and pass through the point Q (x_1, y_1). Let P (x, y) be any point on the line at a distance r from Q (x_1, y_1) i.e. PQ = r. draw PL $\perp$ OX, QM $\perp$ OX, and QN $\perp$ PL. Then

QN = ML = OL – OM = $x - x_1$

and PN = PL – NL = PL – QM = $y - y_1$.



From Δ PQN, $\cos\theta = \dfrac{QN}{PQ} = \dfrac{x - x_1}{r}$

$\Rightarrow \quad \cos\theta = \dfrac{x - x_1}{r}$ ....(1)

Again $\sin\theta = \dfrac{PN}{PQ} \Rightarrow \sin\theta = \dfrac{y - y_1}{r}$ ...(2)

From (1) and (2), we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the required equation of the line in the distance or parametric form.

Note: (i) The equation of the line is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$

$\therefore \quad x - x_1 = r\cos\theta$ and $y - y_1 = r\sin\theta$

$\Rightarrow \quad x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$ is any general point on the line.

(ii) If point P is on right side of Q then r is positive and co-ordinates of P are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and if point P is on left side of Q then r is negative and co-ordinates of P are $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

Illustration 8

A straight line is drawn through the point P (2, 3) and is inclined at an angle of 30°with the x-axis. Find the coordinates of two point on it at a distance 4 from P on either side of P.

Here $(x_1, y_1) = (2, 3)$, $\theta = 30°$. So the equation of the line is

$$\frac{x-2}{\cos 30^\circ} = \frac{y-3}{\sin 30^\circ} \quad \text{or} \quad \frac{x-2}{\frac{\sqrt{3}}{2}} = \frac{y-3}{\frac{1}{2}}$$

or $\quad x - 2 = \sqrt{3}\,(y-3)$ or $x - \sqrt{3}\,y = 2 - 3\sqrt{3}$

point on the line on either side at a distance 4 from P (2, 3) are

$(x_1 \pm r\cos\theta,\ y_1 \pm r\sin\theta)$ or $(2 \pm 4\cos 30°,\ 2 \pm 4\sin 30°)$

or $\quad (2 \pm 2\sqrt{3}, 2 \pm 2)$ or $(2 + 2\sqrt{3}, 4)$ and $(2 - 2\sqrt{3}, 0)$.

Illustration 9

Find the equation of the line through the point A (2, 3) and making an angle of 45° with the x-axis. Also, determine the length of intercept on it between A and the line x + y + 1 = 0.

The equation of a line through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ}$$

or $$\frac{x-2}{1/\sqrt{2}} = \frac{y-3}{1/\sqrt{2}}$$

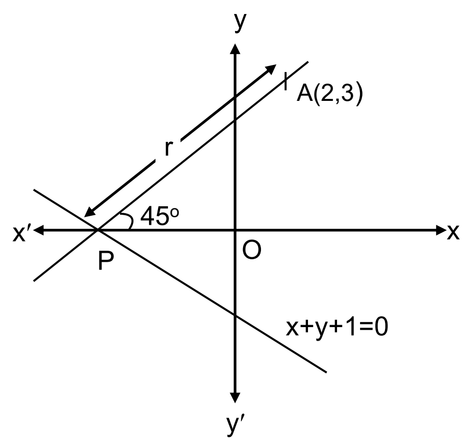
or $\quad x - y + 1 = 0$

Suppose the line meet the line x + y +1 = 0 at P such that AP = r. then the coordinates of P are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

$\Rightarrow \quad x = 2 + r\cos 45^\circ,\ y = 3 + r\sin 45^\circ$

$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$



Thus, the coordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$.

Since P lies on x + y + 1 = 0, So $2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$

$\Rightarrow \quad \sqrt{2}\,r = -6 \Rightarrow r = -3\sqrt{2} \Rightarrow$ length $AP = |r| = 3\sqrt{2}$.

Thus, the length of the intercept $= 3\sqrt{2}$

Transformation of general equation in different standard forms

The general equation of a straight line is Ax + By + C = 0 which can be transformed to various standard forms as discussed below:

Transformation of Ax + By + C = 0 in the slope intercept form (y = mx +c)

We have Ax + By + C = 0 $\Rightarrow$ By = – Ax – C

$$\Rightarrow \quad y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

which is of form y = mx + c, where $m = -\frac{A}{B}$, $c = -\frac{C}{B}$.

Thus, for the straight line Ax + By + C = 0,

$$m = \text{slope} = -\frac{A}{B} = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y} \quad \text{and} \quad \text{Intercept on y-axis} = -\frac{C}{B} = -\frac{\text{Const. term}}{\text{Coeff. of } y}$$

Note : To determine the slope of a line by the formula $m = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$ first transfer all terms in the equation on one side.

Transformation of Ax + By + C = 0 in intercept form $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$

We have Ax + By + C = 0

$$\Rightarrow \quad Ax + By = -C$$

$$\Rightarrow \quad \frac{Ax}{-C} + \frac{By}{-C} = 1$$

$$\Rightarrow \quad \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$$

Which is of the form $\frac{x}{a} + \frac{y}{b} = 1$. Thus, for the straight line Ax + By + C = 0,

Intercept on x-axis $= -\frac{C}{A} = -\frac{\text{const. term}}{\text{coeff. of } x}$

Intercept on y-axis $= -\frac{C}{B} = -\frac{\text{const. term}}{\text{coef. of } y}$

Note : As discussed above the intercepts made by a line with the coordinate axes can be determined by reducing its equation to intercept form. We can also use the following method to determine the intercepts on the axes:
For intercept on x-axis, put y = 0 in the equation of the line and find the value of x. Similarly to find y-intercept, put x = 0 in the equation of the line and find the value of y.

Transformation of Ax + By + C = 0 in the normal form

(x cosα + y sin α = p).

We have Ax + By + C = 0 ...(i)

Let $x \cos\alpha + y \sin\alpha - p = 0$ ...(ii)

be the normal form of Ax + By + C = 0. Then (i) and (ii) represent the same straight line.

$$\therefore \quad \frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{C}{-p}$$

$$\Rightarrow \quad \cos\alpha = -\frac{Ap}{C} \quad \text{and} \quad \sin\alpha = -\frac{Bp}{C} \qquad \text{...(iii)}$$

$\Rightarrow \quad \cos^2\alpha + \sin^2\alpha = \dfrac{A^2p^2}{C^2} + \dfrac{B^2p^2}{C^2}$

$\Rightarrow \quad 1 = \dfrac{p^2}{C^2}(A^2 + B^2) \Rightarrow p = \pm \dfrac{C}{\sqrt{A^2 + B^2}}$

But p denotes the length of the perpendicular form the origin to the line and is always positive.

$\therefore \quad p = \dfrac{C}{\sqrt{A^2 + B^2}}$

Putting the value of p in (iii), we get

$$\cos\alpha = -\frac{A}{\sqrt{A^2 + B^2}}, \ \sin\alpha = -\frac{B}{\sqrt{A^2 + B^2}}$$

So, equation (ii) takes the form

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y - \frac{C}{\sqrt{A^2 + B^2}} = 0$$

$$\Rightarrow \quad -\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}$$

This is the required normal form of the line Ax + By + C = 0

Algorithm to transform the general equation to normal form

Step 1 : Shift the constant term on the RHS and make it positive

Step 2 : Divide both sides by $\sqrt{(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2}$.

The equation so obtained is in the normal form.

Illustration 10

Transform the equation of the line $\sqrt{3}x + y - 8 = 0$ to (i) slope intercept form and find its slope and y-intercept (ii) intercept form and find intercepts on the coordinate axes (iii) normal form and find the inclination of the perpendicular segment form the origin on the line with the axis and its length.

(i) We have

$$\sqrt{3}x + y - 8 = 0 \Rightarrow y = -\sqrt{3}x + 8$$

This is the slope intercept form of the given line.

$\therefore$ slope = $-\sqrt{3}$ and y-intercept = 8

(ii) We have $\sqrt{3}x + y - 8 = 0 \Rightarrow \dfrac{x}{8/\sqrt{3}} + \dfrac{y}{8} = 1$

This is the intercept form of the given line.

So, x-intercept = $\dfrac{8}{\sqrt{3}}$, y-intercept = 8

(iii) We have $\sqrt{3}x + y - 8 = 0 \Rightarrow \sqrt{3}x + y = 8$

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2+1^2}}x+\frac{1}{\sqrt{(\sqrt{3})^2+1^2}}y=\frac{8}{\sqrt{(\sqrt{3})^2+1^2}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}x+\frac{1}{2}y=4$$

This is the normal form of the given line. So,

$$\cos\alpha = \frac{\sqrt{3}}{2},\ \sin\alpha=\frac{1}{2} \text{ and } p=4$$

Since $\sin\alpha$ and $\cos\alpha$ both are positive, therefore α is in first quadrant is equal to $\frac{\pi}{6}$. Hence $\alpha = \frac{\pi}{6}$ and $p = 4$.

Illustration 11

Reduce the lines 3x – 4y + 4 = 0 and 4x – 3y + 12 = 0 to the normal form and hence determine which line is nearer to the origin.

We have $3x - 4y + 4 = 0 \Rightarrow -3x + 4y = 4$

$$\Rightarrow \quad -\frac{3x}{\sqrt{(-3)^2+4^2}}+\frac{4y}{\sqrt{(-3)^2+4^2}}=\frac{4}{\sqrt{(-3)^2+4^2}}$$

$$\Rightarrow \quad -\frac{3}{5}x+\frac{4}{5}y=\frac{4}{5}$$

This is normal form of $3x - 4y + 4 = 0$ and the length of the perpendicular from the origin to it is $p_1 = \frac{4}{5}$.

Now, $4x - 3y + 12 = 0 \Rightarrow -4x + 3y = 12$

$$\Rightarrow \quad -\frac{4x}{\sqrt{(-4)^2+3^2}}+\frac{3y}{\sqrt{(-4)^2+3^2}}=\frac{12}{\sqrt{(-4)^2+3^2}}$$

$$\Rightarrow \quad -\frac{4}{5}x+\frac{3}{5}y=\frac{12}{5}.$$

This is the normal form of $4x - 3y + 12 = 0$ and the length of the perpendicular from the origin to it is $p_2 = \frac{12}{5}$.

Clearly $p_2 > p_1$, therefore the line $3x - 4y + 4 = 0$ is nearer to the origin.

Condition of concurrency of three lines

Three or more lines are said to be concurrent if they pass through a common point i.e., they meet at a point. Thus, if three lines are concurrent the point of intersection of two lines lies on the third line.

Let $a_1x + b_1y + c_1 = 0$ ...(1)

$a_2x + b_2y + c_2 = 0$ ...(2)

$a_3x + b_3y + c_3 = 0$ ...(3)

be three concurrent lines, then the point of intersection of (1) and (2) must lie on the third.

The coordinates of the point of intersection of (1) and (2) are

$$\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$$

This point lies on (3), therefore

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0$$

$\Rightarrow \quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of there lines.

Illustration 12

Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$, are concurrent.

The given lines are concurrent if

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) = 0 \Rightarrow -\lambda - 7 = 0 \Rightarrow \lambda = -7$

ALITER . The given equations are

$3x - 4y - 13 = 0 \quad$ ...(i)

$8x - 11y - 33 = 0 \quad$ ...(ii)

$2x - 3y + \lambda = 0 \quad$ ...(iii)

Solving (i) and (ii) we get $x = 11$ and $y = 5$.

Thus, (11, 5) is the point of intersection of (i) and (ii). The given lines will be concurrent if they pass through the common point i.e. the point of intersection of any two lies on the third. Therefore, (11,5) lies on (iii) i.e.

$2 \times 11 - 3 \times 5 + \lambda = 0 \Rightarrow \lambda = -7$.

Another condition of concurrency of three lines

Three lines

$L_1 = a_1x + b_1y + c_1 = 0;$

$L_2 = a_2x + b_2y + c_2 = 0;$

$L_3 = a_3x + b_3y + c_3 = 0;$

are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero such that

$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$

i.e. $\quad \lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$

Illustration 13

Show that the following lines are concurrent :

$L_1 \equiv (a - b)x + (b - c)y + (c - a) = 0$

$L_2 \equiv (b - c)x + (c - a)y + (a - b) = 0$

and $\quad L_3 \equiv (c - a)x + (a - b)y + (b - c) = 0$

We have $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$,

Where $\lambda_1 = \lambda_2 = \lambda_3 = 1$

Hence the given lines are concurrent.

Line parallel and perpendicular to a given line

Equation of a line parallel to a given line

Prove that the equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.

PROOF. Let m be the slope of the line $ax + by + c = 0$. Then

$$m = -\frac{a}{b} \quad \left(\text{Using } m = -\frac{\text{coeff. of } x}{\text{coeff. of } y}\right)$$

Since the required line is parallel to the given line, the slope of the required line is also m.

Let c_1 be the y-intercept of the required line. Then its equation is

$$y = mx + c_1 \quad \Rightarrow y = -\frac{a}{b}x + c_1 \quad \Rightarrow ax + by - bc_1 = 0$$

$$\Rightarrow \quad ax + by + \lambda = 0, \text{ where } \lambda = -bc_1 = \text{ constant.}$$

Note: To write a line parallel to a given line we keep the expression containing x and y same and simply replace the given constant by a new constant λ. The value of λ can be determined by some given condition.

Equation of a line perpendicular to a given line

Prove that the equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is constant

PROOF: Let $m_1 = -\frac{a}{b}$ and $m_1m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1} = \frac{b}{a}$

Let, c_2 be the y-intercept of the required line. Then its equation is

$$y = m_2x + c_2 \quad \Rightarrow y = \frac{b}{a}x + c_2 \quad \Rightarrow bx - ay + ac_2 = 0$$

$$\Rightarrow \quad bx - ay + \lambda = 0, \text{ where } \lambda = ac_2 = \text{constant.}$$

Angle between two straight lines when their equations are given

The angle θ between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

PROOF: Let m_1 and m_2 be the slope of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Then,

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2}.$$

$$\Rightarrow \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \quad \Rightarrow \tan\theta = \left|\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right| \quad \Rightarrow \theta = \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

Condition for the lines to be parallel

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$m_1 = m_2 \Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

Condition for lines to be perpendicular

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

$$m_1 \; m_2 = -1 \Rightarrow -\frac{a_1}{b_1} \times -\frac{a_2}{b_2} = -1 \Rightarrow a_1a_2 + b_1b_2 = 0$$

Condition for two lines to be Coincident, parallel, perpendicular or intersecting

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

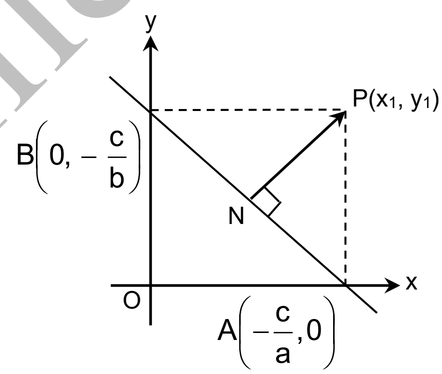
(ii) Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(iv) Perpendicular, if $a_1a_2 + b_1b_2 = 0$

Distance of a point from a line

Prove that the length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$



PROOF: The line $ax + by + c = 0$ meets x-axis at $y = 0$. Therefore putting $y = 0$ in $ax + by + c = 0$, we get $x = -\frac{c}{a}$.

Thus, the coordinates of the point A where the line $ax + by + c = 0$ meet x-axis are $\left(-\frac{c}{a}, 0\right)$. Similarly the coordinates of B are $\left(0, -\frac{c}{b}\right)$.

Let P (x_1, y_1) be the point. Draw PN $\perp$ AB.

Now area of Δ PAB

$$= \frac{1}{2}\left| x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right| = \left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| \quad \text{...(i)}$$

Also, area of Δ PAB$= \frac{1}{2}$ AB.PN $= \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}}$. PN $= \frac{c}{2ab}\sqrt{a^2 + b^2}$. PN ....(ii)

From (i) and (ii) we get

$$= \left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| = \frac{c}{2ab}\sqrt{a^2 + b^2} \text{ . PN}$$

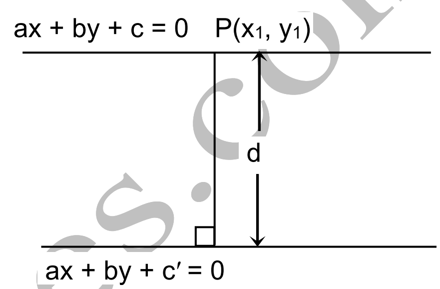
$\Rightarrow \quad PN = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

COR . The length of the perpendicular from the origin to the line

$ax + by + c = 0$ is $\dfrac{|c|}{\sqrt{a^2 + b^2}}$.

Distance between two parallel lines

Let ax + by + c = 0 and ax + by + c′ = 0 be two parallel lines.
Let point P(x_1, y_1) be on line ax + by + c = 0.
Then distance between these two lines is equal to the perpendicular distance from P(x_1, y_1) to the line ax + by + c′ = 0



$\therefore \quad d = \dfrac{|ax_1 + by_1 + c'|}{\sqrt{a^2 + b^2}}$ ......(1)

Also P(x_1, y_1) lies on line ax + by + c = 0

$\therefore \quad ax_1 + by_1 = -c$

$\therefore$ from (1) $d = \dfrac{|-c + c'|}{\sqrt{a^2 + b^2}}$

or $\quad d = \dfrac{|c - c'|}{\sqrt{a^2 + b^2}}$

Note :

Equations of lines parallel to the lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ and dividing distance between the in the ratio m : n internally or externally are :

$$ax + by + \left(\frac{mc_2 + nc_1}{m + n}\right) = 0 \text{ and } ax + by + \left(\frac{mc_2 - nc_1}{m - n}\right) = 0 \text{ respectively.}$$

Illustration 14

If p is the length of the perpendicular from the origin to the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, then prove that $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$

The given line is bx + ay – ab = 0 ...(i)

p = length of the perpendicular from the origin to (i)

$= \dfrac{|b(0) + a(0) - ab|}{\sqrt{b^2 + a^2}} = \dfrac{ab}{\sqrt{a^2 + b^2}}$

$\Rightarrow \quad p^2 = \dfrac{a^2b^2}{a^2 + b^2} \quad \Rightarrow \dfrac{1}{p^2} = \dfrac{a^2 + b^2}{a^2b^2}$

$\Rightarrow \quad \dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$.

Illustration 15

Find the equations of the line parallel to $5x - 12y + 26 = 0$ and at a distance of 4 units from it.

Equation of any line parallel to $5x - 12y + 26 = 0$ is

$$5x - 12y + \lambda = 0 \qquad ....(1)$$

Since the distance between the parallel lines is 4 units, then

$$\frac{|\lambda - 26|}{\sqrt{(5)^2 + (-12)^2}} = 4$$

or $\quad |\lambda - 26| = 52$

or $\quad \lambda - 26 = \pm 52$

or $\quad \lambda = 26 \pm 52$

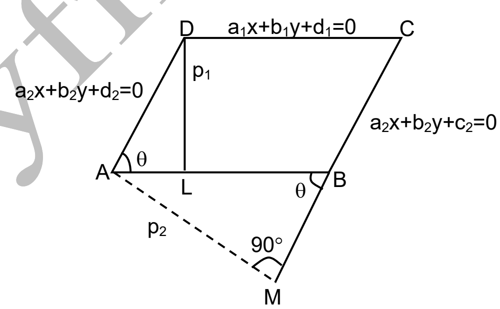
$\therefore \quad \lambda = -26$ or 78

Substituting the values of λ in (1), we get

$5x - 12y - 26 = 0$ and $5x - 12y + 78 = 0$

Area of a parallelogram

Let ABCD be a parallelogram the equations of whose sides AB, BC, CD and DA are $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$. Let p_1 and p_2 be the distance between the pairs of parallel sides of the parallelogram. Then,



$$\sin\theta = \frac{p_1}{AD} \text{ also, } \sin\theta = \frac{p_2}{AB}$$

$$\Rightarrow \quad AD = \frac{p_1}{\sin\theta} \text{ and } AB = \frac{p_2}{\sin\theta}$$

Now,

Area of parallelogram ABCD

$= AB \times p_1$

$= \dfrac{p_1p_2}{\sin\theta} \qquad \left[\because AB = \dfrac{p_2}{\sin\theta}\right]$

Also,

Area of parallelogram ABCD

$= AD \times p_2$

$= \dfrac{p_1p_2}{\sin\theta} \qquad \left[\because AD = \dfrac{p_1}{\sin\theta}\right]$

Thus, area of a parallelogram is $\dfrac{p_1p_2}{\sin\theta}$, where p_1 and p_2 are the distances between parallel sides and θ is the angle between two adjacent sides.

Now,

m_1= slope of AB $= -\dfrac{a_1}{b_1}$

and,

m_2 = Slope of AD = $-\dfrac{a_2}{b_2}$

Since θ is the angle between AB and AD. Therefore,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \quad \tan\theta = \frac{-\dfrac{a_1}{b_1} + \dfrac{a_2}{b_2}}{1 + \dfrac{a_1 a_2}{b_1 b_2}}$$

$$\Rightarrow \quad \tan\theta = \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}$$

$$\Rightarrow \quad \sin\theta = \frac{a_2 b_1 - a_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

We have,

p_1= Distance between parallel sides AB and CD

$$= \frac{|c_1 - d_1|}{\sqrt{a_1^2 + b_1^2}}$$

p_2 = Distance between parallel sides AD and BC

$$= \frac{|c_2 - d_2|}{\sqrt{a_2^2 + b_2^2}}$$

$\therefore$ Area of parallelogram ABCD

$$= \frac{p_1 p_2}{\sin\theta}$$

$$= \frac{|c_1 - d_1||c_2 - d_2|}{|a_2 b_1 - a_1 b_2|}$$

$$= \left| \frac{(c_1 - d_1)(c_2 - d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \right|$$

Illustration 16

Show that the area of the parallelogram formed by the lines $2x - 3y + a = 0$, $3x - 2y - a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y - 2a$ is $\dfrac{2a^2}{5}$ sq. units.

Required area = $\left| \dfrac{(3a - a)\{-2a - (-a)\}}{\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix}} \right| = \dfrac{2a^2}{5}$ sq. units.

Illustration 17

Prove that the four straight lines $\frac{x}{a}+\frac{y}{b}=1, \frac{x}{a}+\frac{y}{b}=2, \frac{x}{b}+\frac{y}{a}=1$ and $\frac{x}{b}+\frac{y}{a}=2$ form a rhombus. Find its area.

The equations of the four sides are

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{....(i)}$$

$$\frac{x}{b}+\frac{y}{a}=1 \quad \text{....(ii)}$$

$$\frac{x}{a}+\frac{y}{b}=2 \quad \text{....(iii)}$$

$$\frac{x}{b}+\frac{y}{a}=2 \quad \text{....(iv)}$$

Clearly, (i), (iii) and (ii), (iv) form two sets of parallel lines. So, the four lines form a parallelogram.

Let p_1 be the distance between parallel lines (i) and (iii) and p_2 be the distance between (ii) and (iv), Then,

$$p_1 = \left|\frac{2-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{ab}{\sqrt{a^2+b^2}} \qquad \left[\text{Using}: d = \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|\right]$$

and,

$$p_2 = \left|\frac{2-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{ab}{\sqrt{a^2+b^2}}$$

Clearly, $p_1 = p_2$. So, the given lines form a rhombus.

$$\text{Area of the rhombus} = \left|\frac{(2-1)(2-1)}{\begin{vmatrix}\frac{1}{a} & \frac{1}{b}\\ \frac{1}{b} & \frac{1}{a}\end{vmatrix}}\right| = \left|\frac{1}{\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}\right| = \frac{a^2b^2}{|b^2-a^2|}$$

PRACTICE ASSIGNMENT II

1. Reduce the following equations into normal from. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis

(i) $x-\sqrt{3}\,y+8=0$ (ii) $y-2=0$ (iii) $x-y=4$

2. (i) Find the equation of the straight line upon which the length of the perpendicular form the origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with +ve direction of X–axis.

(ii) Find the equation of the straight lines which makes a triangle of area of $96\sqrt{3}$ with the axes and perpendicular from the origin to it, makes an angle of 30^0 with y–axis

3. Find the co–ordinates of the point where the line $\sqrt{3}x + y - 8 = 0$ meets the coordinate axes and also find the length of the perpendicular from the origin upon this line and the angle which this perpendicular makes with X–axis

4. Find the equation of the line passing through (2, –1) and

(i) parallel to $2x + 3y = 5$

(ii) perpendicular to $3x + 4y - 7 = 0$

5. Find the equation of the straight line which passes through the point (3,2) and whose gradient is 3/4 Find the co–ordinates of the point on the line that are 5 units from the point (3,2)

6. If the straight line passing through the point P(3,4) makes an angle of $\pi/6$ with the X–axis and meets the line $12x + 5y + 10 = 0$ at Q, find the length PQ

7. The points P(1,1) is translated parallel to the line $2x = y$ in the first quadrant through a unit distance. Show that the co–ordinates of the final point are $\left(1\pm\frac{1}{\sqrt{5}}, 1\pm\frac{2}{\sqrt{5}}\right)$

8. (i) Find the direction in which straight line must be drawn through the point (1,2) so that its point of intersection with line $x + y = 4$ may be at a distance of $\sqrt{6}/3$ from this point

(ii) Find the direction in which a straight line must be drawn through the point (–1, 2) so that ts point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

9. Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$.

10. (i) Find for what value of k the three lines $x - y + 3 = 0$, $x + y = 1$, and $y = kx + 12$ will be concurrent. What is the point of concurrence?

(ii) Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

(iii) If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ and concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.

11. Prove that the lines $3x - 4y + 5 = 0$, $7x - 8y + 5 = 0$ and $4x + 5y = 45$ are concurrent.

12. (i) Let the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable straight line be zero, then show that the line passes through a fixed point whose co-ordinates are (1, 1).

(ii) If the sum of the perpendicular distances of a variable point P(x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10, show that P must move on a line.

13. Find the co-ordinates of the points on the line $x + 5y = 13$ at a distance of 2 units from the line $12x - 5y + 26 = 0$.

14. Find all the points on the line $x + y = 4$ that lie at unit distance from the line $4x + 3y = 10$.

15. (i) If p and p′ are the length of the perpendiculars from the origin to the lines $x \sec\theta - y \operatorname{cosec}\theta = a$ and $x \cos\theta - y \sin\theta = a \cos 2\theta$ respectively then prove that $4p^2 + p'^2 = a^2$

(ii) If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

16. Find the distance between the lines

(i) $3x + 4y = 9$ and $6x + 8y + 15 = 0$

(ii) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$

(iii) $\ell(x + y) + p = 0$ and $\ell(x + y) - r = 0$

17. The three sides of the square are $2x - y - 4 = 0$, $x + 2y - 7 = 0$ and $x + 2y + 3 = 0$
Find the equation of the fourth side

18. Show that the area of the parallelogram formed by the lines
$3y - 2x = a$, $2y - 3x + a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y = 2a$ is $2a^2/5$

19. Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $\dfrac{2c^2}{|ab|}$

20. Find the co–ordinates of those points on the line 3x + 2y = 5 which are equidistant from the lines 4x + 3y = 7 and y = 5/2

21. The equation of two sides of a square are 5x – 12y – 65 = 0 and 5x – 12y + 26 = 0. Find the area of the square.

22. Show that the path of a moving point such that its distances from two lines 3x – 2y = 5 and 3x + 2y = 5 are equal is a straight line.

23. Find the equation of the line which is equidistant from parallel lines 9x + 6y – 7 = 0 and 3x + 2y + 6 = 0

24. Find the equation of lines parallel to 3x – 4y – 5 = 0 at a unit distance from it.

25. Prove that the product of the lengths of the perpendiculars drawn from the points $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ to the line $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ is b^2.

26. A person standing at the junction (crossing) of two straight paths represented by the equations 2x – 3y + 4 = 0 and 3x + 4y – 5 = 0 wants to reach the path whose equation is 6x – 7y + 8 = 0 in the least time. Find equation of the path that he should follow.

Positions of points relative to a line

To find the condition for two points to lie on the same or opposite side of a given line

Let the equation of the given line be ax + by + c = 0 ..(i)

and let the coordinates of the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The coordinates of the point R which divides the line joining P and Q in the ratio m : n are

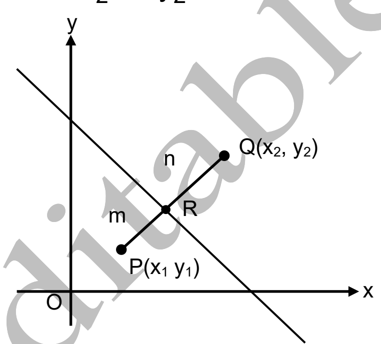
$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) \quad \text{...(ii)}$$

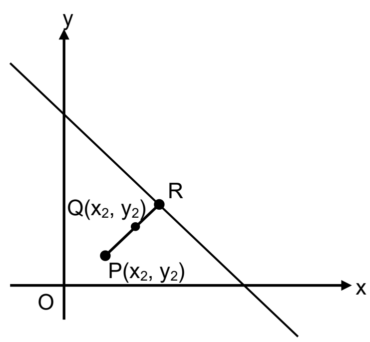
If this points lies on (i), then

$$a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c = 0$$

or, $m(ax_2 + by_2 + c) + n(ax_1 + by_1 + c) = 0$

$$\Rightarrow \quad \frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} \quad \text{...(iii)}$$





If the point R is between the points P and Q i.e., points P and Q are on the opposite sides of the given line, then the ratio m : n is positive.

So from (iii)

$$-\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0 \Rightarrow \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs

If the point R is not between P and Q i.e., point P and Q are on the same side of the given line, then the ratio m : n is negative.
So from (iii)

$$-\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0 \Rightarrow \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign.
Thus, the two points. (x_1, y_1) and (x_2, y_2) are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according as the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

Illustration 18

Are the points (3, 4) and (2, –6) on the same or opposite sides of the 3x – 4y = 8 ?
Let $Z = 3x - 4y - 8$ then Z at (3, 4) is $3 \times 3 - 4 \times 4 - 8 = Z_1$(say)
$\Rightarrow$ $Z_1 = -15$
and Z at (2, – 6) is given by $3 \times 2 - 4 \times - 6 - 8 = Z_2$(say)
$\Rightarrow$ $Z_2 = 6 + 24 - 8 = 22$.
Since Z_1 and Z_2 are of opposite signs, therefore the two points are on the opposite sides of the given line.

EQUATION OF STRAIGHT LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A GIVEN LINE

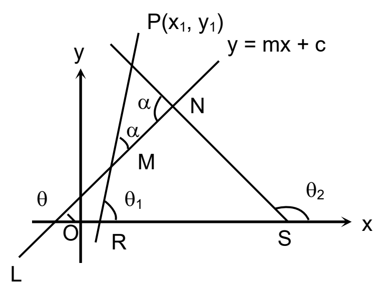
The equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

PROOF Let $P(x_1, y_1)$ be the given point and let the given line be LMN, making an angle θ with the axis of x. Then $m = \tan\theta$.
Let PMR and PNS be two required lines which make angle α with the given line. Let these lines meet the axis of X at R and S respectively. Suppose lines PMR and PNS make angles θ_1 and θ_2 with the positive direction of X-axis. Then the equations of the two required lines are

$$y - y_1 = \tan\theta_1 (x - x_1) \quad .....(1)$$

and $$y - y_1 = \tan\theta_2 (x - x_1) \quad .....(2)$$



In Δ LMR, we have $\theta_1 = \theta + \alpha$
In Δ LNS, we have $\theta_2 = \theta + 180 - \alpha$.
Now, $\theta_1 = \theta + \alpha$

$$\tan\theta_1 = \tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{m + \tan\alpha}{1 - m\tan\alpha}$$

And $\theta_2 = \theta + 180° - \alpha$
$\Rightarrow \tan\theta_2 = \tan(180 + \theta - \alpha) = \tan(\theta - \alpha)$

$$= \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{m - \tan\alpha}{1 + m\tan\alpha}$$

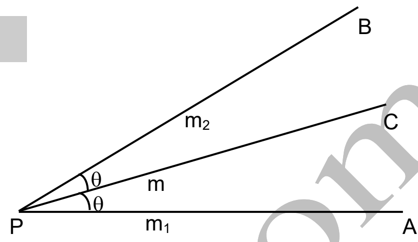
On substituting the values of $\tan\theta_1$ and $\tan\theta_2$ in (1) and (2)
We get

$$y - y_1 = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x_1)$$

and $y - y_1 = \dfrac{m - \tan\alpha}{1 + m\tan\alpha}(x - x_1)$

These are the equations of the two required lines.

Slope of a line equally inclined to two given lines



Let there be two lines with slopes m_1 and m_2 intersecting at a point P. Let there be a line having slope m which makes equal angle θ with the two lines. Then,

$$\tan\theta = \frac{m - m_1}{1 + mm_1}$$

[Taking θ as the angle between PA and PC]

Also,

$$\tan\theta = \frac{m_2 - m}{1 + mm_2} \qquad \text{[Taking } \theta \text{ as the angle between PC and PB]}$$

$$\therefore \quad \frac{m - m_1}{1 + mm_1} = \frac{m_2 - m}{1 + mm_2}$$

Thus, the slope m of a line which is equally inclined with two lines of slopes m_1 and m_2 is given by

$$\frac{m - m_1}{1 + mm_1} = \frac{m_2 - m}{1 + mm_2}$$

REMARK : The above equation in m always gives two values of m which are the slopes of the lines parallel to the bisectors of the angles between the two given lines.

Illustration 19

Find the equations of the two straight lines through (7, 9) and making an angle of 60° with the line $x - \sqrt{3}y - 2\sqrt{3} = 0$.

We know that the equations of two straight lines which pass through a point (x_1, y_1) and make a given angle of α with the given straight line y = mx + c are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Here $x_1 = 7$, $y_1 = 9$, $\alpha = 60°$ and m = slope of the line $x - \sqrt{3}y - 2\sqrt{3} = 0$

So, $m = \dfrac{1}{\sqrt{3}}$

So, the equations of the required lines are

$$y - 9 = \frac{\frac{1}{\sqrt{3}} + \tan 60^\circ}{1 - \frac{1}{\sqrt{3}}\tan 60^\circ}(x - 7)$$

And $$y - 9 = \frac{\frac{1}{\sqrt{3}} - \tan 60^\circ}{1 + \frac{1}{\sqrt{3}}\tan 60^\circ}(x - 7)$$

or $\quad (y-9)\left(1-\frac{1}{\sqrt{3}}\tan 60^\circ\right)=\left(\frac{1}{\sqrt{3}}+\tan 60^\circ\right)(x-7)$

and $\quad (y-9)\left(1+\frac{1}{\sqrt{3}}\tan 60^\circ\right)=\left(\frac{1}{\sqrt{3}}-\tan 60^\circ\right)(x-7)$

or $\quad 0=\left(\frac{1}{\sqrt{3}}+\sqrt{3}\right)(x-7) \Rightarrow x-7=0$ and $(y-9)(2)=\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right)(x-7)$

$\Rightarrow \quad x+\sqrt{3}\,y=7+9\sqrt{3}$

Hence the required lines are $x = 7$ and $x+\sqrt{3}\,y=7+9\sqrt{3}$

Illustration 20

Show that the equations of the straight lines passing through the point (3, –2) and inclined at 60° to the line $\sqrt{3}\,x + y = 1$ are $y + 2 = 0$ and $y - \sqrt{3}\,x + 2 + 3\sqrt{3} = 0$

The equations of two straight lines passing through a point (x_1, y_1) and making an angle α with $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Here $\quad x_1 = 3,\ y_1 = -2,\ \alpha = 60^\circ$ and m = slope of $\sqrt{3}\,x + y = 1$. So, $m = -\sqrt{3}$

So, the equations of the required lines are

$$y+2=\frac{-\sqrt{3}-\tan 60^\circ}{1-\sqrt{3}\tan 60^\circ}(x-3)$$

and $$y+2=\frac{-\sqrt{3}+\tan 60^\circ}{1+\sqrt{3}\tan 60^\circ}(x-3)$$

or $\quad y+2=\sqrt{3}(x-3)$ and $y + 2 = 0$

or $\quad y-\sqrt{3}\,x+2+3\sqrt{3}=0$ and $y + 2 = 0$

Family of lines through the intersection of two given lines

That the equation of the family of lines passing through the intersection of the lines
$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is
$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, where λ is a parameter.

PROOF : The equations of the lines are

$a_1x + b_1y + c_1 = 0 \quad$ .....(i)

and, $a_2x + b_2y + c_2 = 0 \quad$ .....(ii)

Let (α, β) be the point of intersection of the lines (i) and (ii),

Then,

$a_1\alpha + b_1\beta + c_1 = 0 \quad$ ....(iii)

and, $a_2\alpha + b_2\beta + c_2 = 0 \quad$ ....(iv)

Now, consider the equation

$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$

Clearly, this is a first degree equation in x and y. So it represents a straight line.

We have,

$(a_1\alpha + b_1\beta + c_1) + \lambda(a_2\alpha + b_2\beta + c_2) = 0 + \lambda 0 = 0 \quad$ [Using (iii) and (iv)]

So, (α, β) lies on the line given in (v).

Hence, (v) represents family lines through the point of intersection of (i) and (ii),
Thus, the family of straight lines through the intersection of lines $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ is

$$(a_1x + b_1y + c_1) + \lambda(a_2 + b_2y + c_2) = 0$$

i.e. $L_1 + \lambda L_2 = 0$

REMARK The equation $L_1 + \lambda L_2 = 0$ represents a line passing through the intersection of the lines $L_1 = 0$ and $L_2 = 0$ which is a fixed point. Hence, $L_1 + \lambda L_2 = 0$ represents a family of straight lines, for different values of λ, which pass through a fixed-point.

Illustration 21

Find the equation of the straight line which passes through the point (2, –3) and the point of intersection of lines x + y + 4 = 0 and 3x – y – 8 = 0.

Any line through the intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ has the equation

$$(x + y + 4) + \lambda (3x - y - 8) = 0 \quad ....(i)$$

This will pass through (2, –3) if

$$(2 - 3 + 4) + \lambda (6 + 3 - 8) = 0$$

$\Rightarrow$ $3 + \lambda = 0$

$\Rightarrow$ $\lambda = -3$

Putting the value of λ in (i), the required line is

$$2x - y - 7 = 0$$

ALITER. Solving the equations $x + y + 4 = 0$ and $3x - y - 8 = 0$ by cross-multiplication, we get $x = 1$, $y = -5$.
So the two lines intersect at the point (1, –5). Hence, the required line passes through (2, –3) and (1, –5) and so its equation is

$$y + 3 = \frac{-5+3}{1-2}(x - 2) \Rightarrow 2x - y - 7 = 0$$

Illustration 22

Prove that all the lines represented by the equation

$$\mathbf{(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y - (5\cos\theta - 2\sin\theta) = 0}$$

pass through a fixed-point for all values of θ.

We have,

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y - (5\cos\theta - 2\sin\theta) = 0$$

$\Rightarrow$ $(2x + 3y - 5)\cos\theta + (3x - 5y + 2)\sin\theta = 0$

$\Rightarrow$ $(2x + 3y - 5) + (3x - 5y + 2)\tan\theta = 0$

Clearly, it represents a family of straight lines, for different values of θ, passing through a fixed-point which is the point of intersection of the lines $2x + 3y - 5 = 0$ and $3x - 5y + 2 = 0$. Solving these two equations, we obtain that the coordinates of the point are (1, 1)

Illustration 23

If 3a + 2b + 6c = 0 and the family of straight lines ax + by + c = 0 passes through a fixed point.
Find the co-ordinates of fixed point.

Given $3a + 2b + 6c = 0$

or $\dfrac{a}{2} + \dfrac{b}{3} + c = 0 \quad ....(1)$

and family of straight lines is

$$ax + by + c = 0 \qquad ....(2)$$

Subtracting (1) from (2) then

$$a\left(x-\frac{1}{2}\right)+b\left(y-\frac{1}{3}\right)=0$$

which is equation of a line passing through the point of intersection of the lines

$$x-\frac{1}{2}=0 \text{ and } y-\frac{1}{3}=0$$

$\therefore$ The co-ordinates of fixed point are $\left(\frac{1}{2},\frac{1}{3}\right)$.

Illustration 24

The family of lines x (a + 2b) + y (a + 3b) = a + b passes through the point for all values of a and b. Find the point.

The given equation can be written as

$$a(x + y - 1) + b(2x + 3y - 1) = 0$$

which is equation of a line passing through the point of intersection of the lines $x + y - 1 = 0$ and $2x + 3y - 1 = 0$. The point of intersection of these lines is (2, –1). Hence the given family of lines passes through the point (2, –1) for all values of a and b.

Illustration 25

If $4a^2 + 9b^2 - c^2 + 12ab = 0$, then the family of straight lines ax + by + c = 0 is either concurrent at …. or at …..

Given $4a^2 + 9b^2 - c^2 + 12ab = 0$

or $(2a + 3b)^2 - c^2 = 0$

or $c = \pm(2a + 3b) \qquad ....(1)$

and family of straight lines is

$$ax + by + c = 0 \qquad ....(2)$$

Substituting the value of c from (1) in (2), then

$$ax + by \pm (2a + 3b) = 0$$

$\Rightarrow$ $a(x \pm 2) + b(y \pm 3) = 0$

Taking '+' sign:

Which is equation of a line passing through the point of intersection of the lines $x + 2 = 0$ and $y + 3 = 0$

$\therefore$ Co-ordinates of fixed point are (–2, –3).

Taking '–' sign:

Which is equation of a line passing through the point of intersection of the lines

$$x - 2 = 0 \text{ and } y - 3 = 0$$

$\therefore$ Co-ordinates of fixed point are (2, 3)

Hence the family of straight lines ax + by + c = 0 is either concurrent at (–2, –3) or at (2, 3).

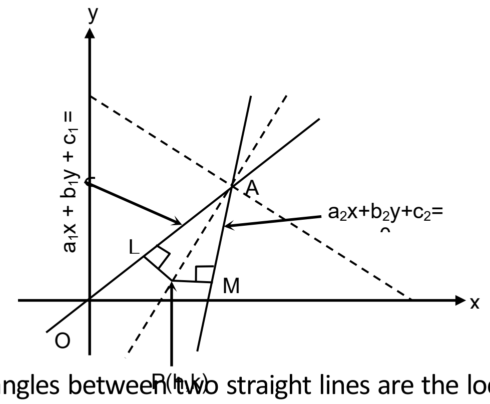
Equations of the bisectors of the angles between two straight lines

Prove that the equation of the bisectors of the angles between the lines

$$a_1x + b_1y + c_1 = 0 \qquad ...(i)$$

and $$a_2x + b_2y + c_2 = 0 \qquad ...(ii)$$

are given by $$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$



PROOF: The bisectors of the angles between two straight lines are the locus of a point which is equidistant from the two lines. So let P (h, k) be a point equidistant from the lines (i) and (ii). Then

$$|PL| = |PM|$$

$$\Rightarrow \quad \left|\frac{a_1h + b_1k + c_1}{\sqrt{a_1^2 + b_1^2}}\right| = \left|\frac{a_2h + b_2k + c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

$\therefore$ Locus of (h, k) i.e., the equations of the bisectors are

$$\frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2x + b_2y + c_2|}{\sqrt{a_2^2 + b_2^2}}$$

$$\Rightarrow \quad \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

These are the required equations of the bisectors.

Illustration 26

Find the equations of the bisectors of the angles between the straight lines
$3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$

The equations of the bisectors of the angles between
$3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$ are

$$\frac{3x - 4y + 7}{\sqrt{3^2 + (-4)^2}} = \pm \frac{12x - 5y - 8}{\sqrt{12^2 + (-5)^2}}$$

or $$\frac{3x - 4y + 7}{5} = \pm \frac{12x - 5y - 8}{13}$$

or $39x - 52y + 91 = \pm (60x - 25y - 40)$

taking the positive sign, we get $21x + 27y - 131 = 0$ as one bisector
taking the negative sign, we get $99x - 77y + 51 = 0$ as the other bisector.

Algorithm to find the bisector of the angle containing the origin

Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. To find the bisector of the angle containing the origin, we proceed as follows:

STEP 1. See whether the constant terms c_1 and c_2 in the equations of two lines are positive or not. If not, then multiply both the sides of the equations by – 1 to make the constant term positive.

STEP 2. Now obtain the bisector corresponding to the positive sign i.e.

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

This is the required bisector of the angle containing the origin.

Note. Note that the bisector of the angle containing the origin means the bisector of that angle between the lines which contains the origin within it.

Algorithm to find the bisectors of acute (internal) and obtuse (external) angles between two lines

Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. To separate the bisectors of the obtuse and acute angles between the lines we proceed as follows:

STEP 1. See whether the constant terms c_1 and c_2 in the two equation are positive or not. If not, then multiply both the sides of the equations by – 1 to make the constant term positive.

STEP 2. Determine the sign of the expression $a_1a_2 + b_1 b_2$

STEP 3. If $a_1a_2 + b_1 b_2 > 0$, then the bisector corresponding to "+" and "–" sign give the obtuse and acute angle bisectors respectively i.e.

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

and
$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

are the bisectors of obtuse and acute angles respectively.

If $a_1a_2 + b_1 b_2 < 0$, then the bisector corresponding to "+" and "–" sign give the acute and obtuse angle bisectors respectively i.e.

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

and
$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

are the bisectors of acute and obtuse angles respectively.

Illustration 27

Find the equation of the obtuse angle bisector of lines 12x – 5y + 7 = 0 and 3y – 4x – 1 = 0

First we make the constant terms (c_1 and c_2) positive in the two equations. Making constant terms positive. The two equations become

$12x – 5y + 7 = 0$ and $4x – 3y + 1 = 0$

Now $a_1a_2 + b_1b_2 = 12 \times 4 + (–5) \times (–3) = 63$ which is positive. Hence, "+" sign gives the obtuse angle bisector. The obtuse angle bisector is

$$\frac{12x-5y+7}{\sqrt{12^2+(-5)^2}}=+\frac{4x-3y+1}{\sqrt{4^2+(-3)^2}}$$

$\Rightarrow$ $5(12x – 5y + 7) = 13 (4x – 3y + 1)$

$\Rightarrow$ $4x + 7y + 11 = 0$, is the required obtuse angle bisector.

Illustration 28

Find the equation of the bisector of the obtuse angle between the lines 3x – 4y + 7 = 0 and 12x + 5y – 2 = 0.

Firstly make the constant terms (c_1, c_2) positive

$3x - 4y + 7 = 0$ and $-12x - 5y + 2 = 0$

$\because \quad a_1a_2 + b_1b_2 = (3)(-12) + (-4)(-5) = -36 + 20 = -16$

$\therefore \quad a_1a_2 + b_1b_2 < 0$

Hence "–" sign gives the obtuse bisector.

$\therefore$ obtuse bisector is $\dfrac{(3x-4y+7)}{\sqrt{(3)^2+(-4)^2}} = -\dfrac{(-12x-5y+2)}{\sqrt{(-12)^2+(-5)^2}}$

$\Rightarrow \quad 13(3x - 4y + 7) = -5(-12x - 5y + 2)$

$\Rightarrow \quad 21x + 77y - 101 = 0$

is the obtuse angle bisector.

Illustration 29

Find the bisector of acute angle between the lines x + y – 3 = 0 and 7x – y + 5 = 0.

Firstly, make the constant terms (c_1,c_2) positive then $-x - y + 3 = 0$ and $7x - y + 5 = 0$

$\because \quad a_1a_2 + b_1b_2 = (-1)(7) + (-1)(-1) = -7 + 1 = -6$

i.e., $\quad a_1a_2 + b_1b_2 < 0$

Hence "+" sign gives the acute angle bisector :

$\therefore$ acute bisector is $\dfrac{-x-y+3}{\sqrt{(-1)^2+(-1)^2}} = +\dfrac{7x-y+5}{\sqrt{(7)^2+(-1)^2}}$

$\Rightarrow \quad -5x - 5y + 15 = 7x - y + 5$

$\Rightarrow \quad 12x + 4y - 10 = 0$ or $6x + 2y - 5 = 0$

is the acute angle bisector.

ALGORITHM TO DETERMINE WHETHER THE ORIGIN LIES IN THE OBTUSE ANGLE OR ANGLE OR ACUTE ANGLE BETWEEN THE LINES

Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. To determine whether the origin lies in the acute angle or obtuse angle between the lines we proceed as follows:

STEP 1. See whether the constant terms c_1 and c_2 in the equations of the two lines are positive or not. If not, make them positive by multiplying both sides of the equations by minus sign.

STEP 2. Determine the sign of $a_1a_2 + b_1b_2$.

STEP 3. If $a_1a_2 + b_1b_2 > 0$, then the origin lies in the obtuse angle and the "+" sign gives the bisector of the obtuse angle. If $a_1a_2 + b_1b_2 < 0$, then the origin lies in the acute angle and "+" sign gives the bisector of acute angle.

Illustration 30

For the straight lines 4x + 3y – 6 = 0 and 5x + 12y + 9 = 0 find the equation of the bisector of the angle which contains the origin.

To find the bisector of the angle between the lines which contains the origin, we first write down the equation of the given lines in such a form that the constant terms in the equations of the lines are positive. The equations of the given lines are

$4x + 3y - 6 = 0 \Rightarrow -4x - 3y + 6 = 0$ ..(i)

and $\quad 5x + 12y + 9 = 0$ (ii)

Now the equation of the bisector of the angle between the lines which contains the origins the bisector corresponding to the positive sign i.e.,

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \qquad \text{(taking +ve sign)}$$

$\Rightarrow \quad -52x - 39y + 78 = 25x + 60y + 45$

$\Rightarrow \quad 7x + 9y - 3 = 0$

From (i) and (ii) we have $a_1a_2 + b_1b_2 = -20 - 36 < 0$

Therefore the origin is situated in an acute angle region and the bisector of this angle is

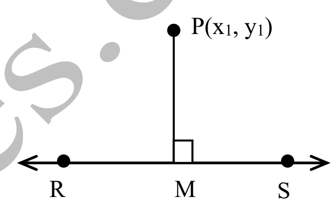
$7x + 9y - 3 = 0$

The foot of perpendicular drawn from the point (x_1, y_1) to the line $ax + by + c = 0$

Method – I

Let P (x_1, y_1) be the point and let M be the foot of perpendicular drawn from P on $ax + by + c = 0$

Then the equation of PM which is perpendicular to RS is $y - y_1 = \frac{b}{a}(x - x_1)$



Solving it with $ax + by + c = 0$ we get M.

Method II

Let M be (x_2, y_2). Then M lies on $ax + by + c = 0$

$\Rightarrow ax_2 + by_2 + c = 0 \qquad .... (1)$

Also PM $\perp$ RS $\Rightarrow$ (slope of PM) (Slope of RS) = –1

$$\Rightarrow \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(\frac{-a}{b}\right) = -1 \qquad ....(2)$$

Solving (1) and (2) we get M.

Method III

From (2) we get $\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b}$

$$\Rightarrow \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{a(x_2 - x_1) + b(y_2 - y_1)}{a^2 + b^2} \quad \text{(By ratio proportion method)}$$

$$\Rightarrow \quad \frac{(ax_2 + by_2) - (ax_1 + by_1)}{a^2 + b^2}$$

$$\Rightarrow \quad \frac{-c - (ax_1 + by_1)}{a^2 + b^2}$$

$$\Rightarrow \quad \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

Illustration 31

Find the coordinates of the foot of perpendicular drawn from the point (2, 3) to the line y = 3x + 4.

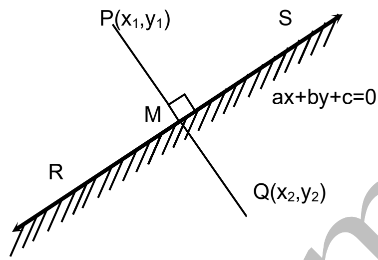
$$\frac{x_2 - 2}{3} = \frac{y_2 - 3}{-1} = \frac{-(3 \times 2 - 3 + 4)}{3^2 + (-1)}$$

$\Rightarrow x_2 = \frac{-1}{10}$, $y_2 = \frac{37}{10}$

$\therefore M \equiv \left(\frac{-1}{10}, \frac{37}{10}\right)$

Image or reflection of a point (x_1, y_1) about a line mirror.

Method – I Let $Q = (x_2, y_2)$ be the image of $P \equiv (x_1, y_1)$ then find co-ordinates of the foot of the perpendicular M draw from the point $P(x_1, y_1)$ on RS and use the fact that M is the mid point of P and Q.



i.e., $M \equiv \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Method – II Ratio Proportion Method :

$\because$ PQ $\perp$ RS

$\therefore$ (slope of PQ)$\times$ (slope of RS) = – 1

or $\left(\frac{y_2-y_1}{x_2-x_1}\right)\times\left(-\frac{a}{b}\right) = -1$ or $\frac{(x_2-x_1)}{a} = \frac{(y_2-y_1)}{b}$

or $\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{a(x_2-x_1)+b(y_2-y_1)}{a^2+b^2}$

$= \frac{a(2x-x_1-x_1)+b(2y-y_1-y_1)}{a^2+b^2}$ $\left(\begin{array}{l}\because M(x, y) \text{ is mid point of P and Q} \\ \therefore x_2 = 2x - x_1 \text{ and } y_2 = 2y - y_1\end{array}\right)$

$= \frac{-2ax_1-2by_1+2(ax+by)}{a^2+b^2}$

$= \frac{-2ax_1-2by_1+2(-c)}{a^2+b^2}$ $(\because$ ax + by = – c)

$= \frac{-2(ax_1+by_1+c)}{(a^2+b^2)}$

i.e., $\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = -\frac{2(ax_1+by_1+c)}{(a^2+b^2)}$

Method III : By distance form or symmetric form or Parametric form :

$\because$ Slope of RS = $-\frac{a}{b}$

$\therefore$ Slope of PQ = $\frac{b}{a}$

Let $\tan\theta = \frac{b}{a}$

$\therefore$ $\sin\theta = \frac{b}{\sqrt{a^2+b^2)}}$ and $\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}}$

Put the equation of the mirror line such that the coefficient of y becomes negative. Suppose if b > 0

then ax + by + c = 0 becomes – ax – by – c = 0

and p = PM

= Directed distance from $P(x_1, y_1)$ on – ax – by – c = 0 (i.e., p +ve or –ve)

$= \left(\frac{-ax_1-by_1-c}{\sqrt{(a^2+b^2)}}\right)$

$\therefore$ PQ = 2PM

$$= 2p$$

$$= 2\left(\frac{-ax_1 - by_1 - c}{\sqrt{(a^2 + b^2)}}\right)$$

$$= r$$

$\Rightarrow$ Required image has the co-ordinates $(x_1 + r\cos\theta, y_1 + r\sin\theta)$.

Illustration 32

Find the image of the point (4, –13) with respect to the line mirror 5x + y + 6 = 0

Let $P \equiv (4, -13)$ and Let $Q \equiv (x_2, y_2)$ be mirror image P with respect to line mirror $5x + y + 6 = 0$. Let M (α, β) be the foot of perpendicular from P (4, –13) on the line mirror $5x + y + 6 = 0$, then

$$\frac{\alpha - 4}{5} = \frac{\beta + 13}{1} = \frac{-(5 \times 4 - 13 + 6)}{5^2 + 1^2}$$

or $$\frac{\alpha - 4}{5} = \frac{\beta + 13}{1} = -\frac{1}{2}$$

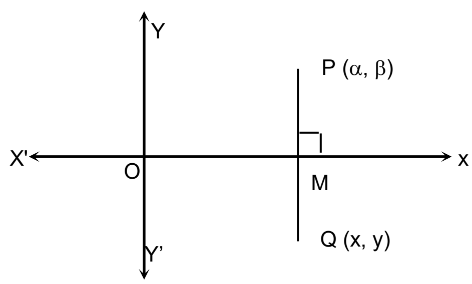
$$M \equiv \left(\frac{3}{2}, -\frac{27}{2}\right)$$

$\because$ M is the mid point of P and Q, then

$$Q \equiv (x_2, y_2) \equiv \left(2 \times \frac{3}{2} - 4, 2 \times \left(-\frac{27}{2}\right) + 13\right)$$

i.e., $Q \equiv (-1, -14)$

Image or reflection of a point in different cases



(i) The image or reflection of a point with respect to x-axis :

Let $P(\alpha, \beta)$ be any point and Q (x, y) be its image about x-axis, then (M is the mid point of P and Q)

$$x = \alpha$$

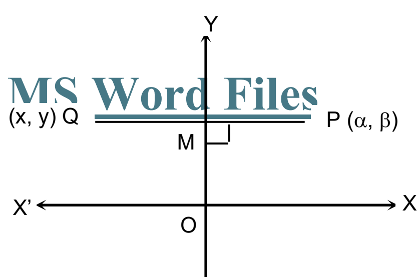
$$y = -\beta$$

$$\therefore \quad Q = (\alpha, -\beta)$$

i.e., sign change of ordinate.

Note: The image of the line $ax + by + c = 0$ about x-axis is $ax - by + c = 0$

(ii) The image or reflection of a point with respect to y-axis:



Let P (α, β) be any point and Q (x, y) be its image about y-axis, then (M is the mid point of PQ)

$x = -\alpha$ and $y = \beta$

$\therefore$ $Q \equiv (-\alpha, \beta)$

i.e., sign change of abscissae.

Note: The image of the line $ax + by + c = 0$ about y-axis is $-ax + by + c = 0$

(iii) The image or reflection of a point with respect to origin :

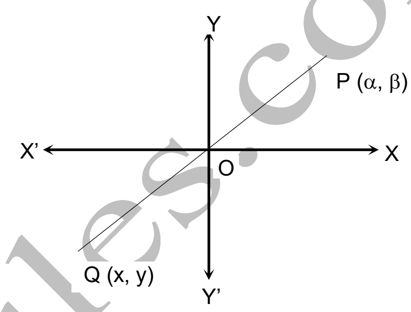
Let P (α, β) be any point and Q (x, y) be its image about the origin (O is the mid point of PQ), then

$x = -\alpha$ and $y = -\beta$

$\therefore$ $Q = (-\alpha, -\beta)$

i.e., sign change of abscissae and ordinate.

Note: The image of the line $ax + by + c = 0$ about origin is $-ax - by + c = 0$



(iv) The image or reflection of a point with respect to the line x = a:

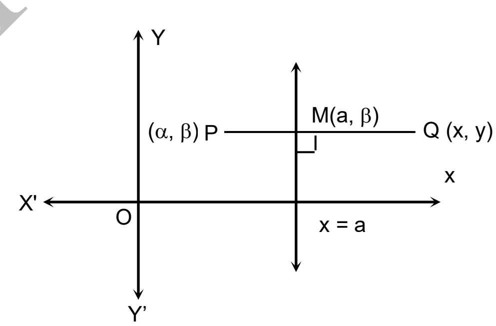
Let P (α, β) be any point and Q (x, y) be its image about the line x = a, then $y = \beta$

$\therefore$ Co-ordinates of M are (a, β)

$\because$ M is the mid point of PQ

$\therefore$ $Q \equiv (2a - \alpha, \beta)$

Note: The image of the line $ax + by + c = 0$ about the line $x = \lambda$ is $a(2\lambda - x) + by + c = 0$



(v) The image or reflection of a point with respect to the line y = b:

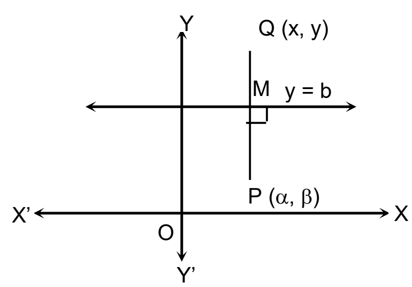
P (α, β) be any point and Q (x, y) be its image about the line y = b, then $x = \alpha$

$\therefore$ Co-ordinates of M are (α, b)

$\because$ M is the mid point of PQ

$\therefore$ $Q \equiv (\alpha, 2b - \beta)$

Note: The image of the line $ax + by + c = 0$ about the line $y = \mu$ is $ax + b(2\mu - y) + c = 0$



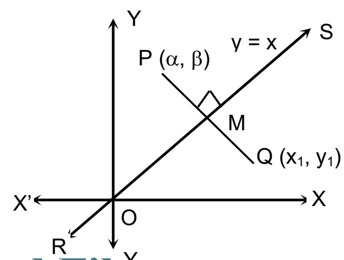
(vi) The image or reflection of a point with respect to the line y = x:

Let P (α, β) be any point and Q (x_1, y_1) be its image about the line y = x (RS), then $PQ \perp RS$

$\therefore$ (Slope of PQ) $\times$ (Slope of RS) = -1

or $\dfrac{y_1 - \beta}{x_1 - \alpha} \times 1 = -1$ or $x_1 - \alpha = \beta - y_1$ ......(1)

and mid point of PQ lie on y = x



$$\left(\frac{y_1+\beta}{2}\right)=\left(\frac{x_1+\alpha}{2}\right)$$

or $x_1 + \alpha = \beta + y_1$ ....(2)

solving (1) and (2), we get $x_1 = \beta$ and $y_1 = \alpha$

$\therefore$ $Q \equiv (\beta, \alpha)$ i.e., interchange of x and y.

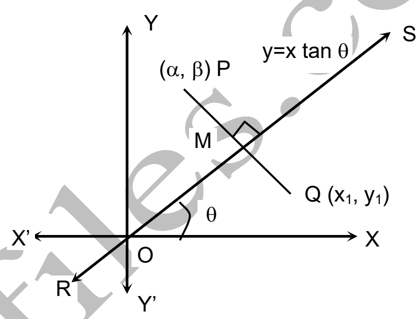
Note: The image of the line $ax + by + c = 0$ about the line $y = x$ is $ay + bx + c= 0$

(vii) The image or reflection of a point with respect to the line $y = x \tan\theta$:

Let $P(\alpha, \beta)$ be any point and $Q(x_1, y_1)$ be its image about the line $y = x \tan\theta$ (RS), then $PQ \perp RS$

$\therefore$ (Slope of PQ) $\times$ (Slope of RS) = -1

or $$\frac{y_1-\beta}{x_1-\alpha}\times\tan\theta=-1$$



$\Rightarrow$ $y_1 - \beta = (\alpha - x_1)\cot\theta$ .....(1)

and mid point of PQ lie on $y = x \tan\theta$

i.e., $$\left(\frac{y_1+\beta}{2}\right)=\left(\frac{x_1+\alpha}{2}\right)\tan\theta$$

or $y_1 + \beta = (x_1 + \alpha)\tan\theta$ ....(2)

solving (1) ad (2), we get

$x_1 = \alpha\cos 2\theta + \beta\sin 2\theta$

and $y_1 = \alpha\sin 2\theta - \beta\cos 2\theta$

$\therefore$ $Q(\alpha\cos 2\theta + \beta\sin 2\theta, \alpha\sin 2\theta - \beta\cos 2\theta)$

Illustration 33

Find the image of the point (–2, –7) under the transformations $(x, y) \to (x - 2y, -3x + y)$.

Let (x_1, y_1) be the image of the point (x, y) under the given transformations.

Then

$x_1 = x - 2y = (-2) -2(-7) = 12$

$\therefore$ $x_1 = 12$ and $y_1 = -3x + y = -3(-2) -7 = -1$

$\therefore$ $y_1 = -1$

Hence the image is (12, -1).

Illustration 34

The point (4, 1) undergoes the following three transformations successively :

(i) Reflection about the line y = x.

(ii) Translation through a distance 2 units along the positive direction of x-axis.

(iii) Rotation through an angle $\pi / 4$ about the origin in the anticlockwise direction. Then find the co-ordinates of the final position.

Let Q (x_1, y_1) be the reflection of P about the line y = x. Then

$$\left.\begin{aligned} x_1 &= 1 \\ y_1 &= 4 \end{aligned}\right\} \quad ....(1)$$

$\therefore$ Co-ordinates of Q is (1, 4).

Given that Q move 2 units along the positive direction of x-axis.

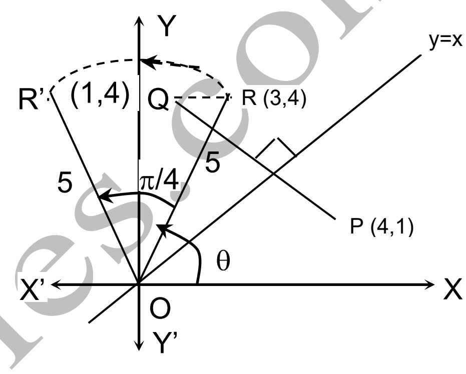
$\therefore$ Co-ordinates of R is $(x_1 + 2, y_1)$

or R (3, 4)

If OR makes an angle θ, then

$$\tan\theta = \frac{4}{3}$$

$$\sin\theta = \frac{4}{5} \text{ and } \cos\theta = \frac{3}{5}$$



After rotation of $\frac{\pi}{4}$ let new position of R is R' and

$$OR = OR' = \sqrt{3^2 + 4^2} = 5$$

$\therefore$ OR' makes an angle $(\pi / 4 + \theta)$ with x-axis.

Co-ordinates of R' $\left(OR'\cos\left(\frac{\pi}{4}+\theta\right), OR'\left(\frac{\pi}{4}+\theta\right)\right)$

i.e., $R'\left(OR'\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right), OR'\sin\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\right)$

$\Rightarrow$ $R'\left(5\left(\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right), 5\left(\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}\right)\right)$

$\Rightarrow$ $R'\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$

PRACTICE ASSIGNMENT III

1. Do the points (3,4) (7,8) lie on same side of line 3x + y – 14 = 0 ?

2. If the point (a, a) falls between the lines | x + y | = 2, find the range of 'a'

3. If the points (4, 7) and $(\cos\theta, \sin\theta)$ where $0 < \theta < \pi$ lie on the same side of line x + y – 1 = 0 then prove that θ lies in the first quadrant.

4. If $P(1+t/\sqrt{2}, 2+t/\sqrt{2})$ lies between the parallel lines x + 2y = 1 and 2x + 4y = 15 then find the range of t

5. (i) Find the equation to the straight line passing through the point (3, –2) and inclined at 60° to the line $\sqrt{3}x + y = 1$.

(ii) Find the equation of the line through the point (3, 2) which make an angle of 45° with the line x – 2y = 3.

6. A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is x + y = 2. Find the equations of the other sides of the triangle.

7. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. Point B and C are chosen on these two lines such that AB = AC. Determine the possible equations of the line BC passing through the point (1, 2)

8. Find the equations to the straight line passing through the point (4,5) and equally inclined to the lines $3x = 4y + 7$ and $5y = 12x + 6$

9. Obtain the equations of the lines passing through the intersection of lines $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and equally inclined to the axes.

10. Prove that the straight line given by the equation $(3x - 4y + 7) + k(5x - 2y - 7) = 0$ passes through a fixed point for all values of k and find the point.

11. Find the equation to the straight-line passing through the point (3, 2) and the point of intersection of the lines $2x + 3y = 1$ and $3x - 4y = 6$.

12. Determine the equation of the straight line which passes through the intersection of the lines given by $3x - 4y + 1 = 0$ and $5x + y = 1$, and has equal intercepts of the same sign along the axes.

13. Find the equation of the straight line passing through the intersection of $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and parallel to $4x + 5y = 6$.

14. Find the equation of the straight line which passes through the intersection of the straight lines $2x + 3y = 5$ and $3x + 5y = 7$ and makes equal positive intercepts on the axes.

15. Find the equation of the straight line which passes through the intersection of the lines $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ and is parallel to
(a) x–axis (b) y-axis (c) $3x + 4y = 14$

16. Find the equation of the straight line which passes through the point of intersection of the straight lines $x + 2y = 5$ and $3x + 7y = 17$ and is perpendicular to the straight line $3x + 4y = 10$

17. Find the bisector of acute angle between the lines $2x - y + 4 = 0$ and $x - 2y = 1$.

18. Find the equation of the bisector of that angle between the lines $x + y = 3$ and $2x - y = 2$ which contains (i) origin (ii) (1,1)

19. Find the equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$

20. Find the equation of the bisector of the angle between the lines $x + 2y - 11 = 0$ and $3x - 6y - 5 = 0$ which contains the point (1, –3).

21. Find the equations of the bisectors L_1 and L_2 between the lines $3x - 4y + 5 = 0$ and $5x + 12y - 1 = 0$. If L_1 meets the coordinate axes in A and B whereas L_2 meets in C and D, then prove that 2AC = DB.

22. Find the co–ordinates of the foot of the perpendicular drawn from the point (2,3) to the line $y = 3x + 4$

23. Find the image of the point (2,4) about the line $2x + y + 6 = 0$

24. If $2a - 5b - 3c = 0$, show that the straight line $ax + by + c = 0$ passes through a fixed point. Find the equation of the straight line passing through this fixed point and parallel to the straight line $9x + 12y = 20$.

25. Show that the straight lines given by $(2 + k)x + (1 + k)y = 5 + 7k$ for different values of k pass through a fixed point. Also, find that point.

Objective Assignment

1. If the lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4cy + c = 0$ are concurrent, then a, b, c are in :
(a) AP **(b)** GP
(c) HP **(d)** AGP

2. The straight line through the point of intersection of $ax + by + c = 0$ and $a'x + b'y + c' = 0$ and parallel to y-axis has the equation :
(a) $x(ab' - a'b) + (cb' - c'b) = 0$ **(b)** $x(ab' + a'b) + (cb' + c'b) = 0$
(c) $y(ab' - a'b) + (c'a - ca') = 0$ **(d)** $y(ab' + a'b) + (c'a + ca') = 0$

3. Find the equation of the line through the intersection of $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and perpendicular to $6x - 7y + c = 0$:

(a) $199y + 120x = 125$ (b) $199y - 120x = 125$
(c) $119x + 102y = 125$ (d) $119x - 102y = 125$

4. If the quadrilateral formed by the lines $ax + by + c = 0$, $a'x + b'y + c = 0$, $ax + by + c' = 0$, $a'x + b'y + c' = 0$ have perpendicular diagonals, then
(a) $b^2 + c^2 = b'^2 + c'^2$ (b) $c^2 + a^2 = c'^2 + a'^2$
(c) $a^2 + b^2 = a'^2 + b'^2$ (d) none of these

5. If the orthocentre of the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 1 = 0$, $ax + by - 1 = 0$ is at the origin, then (a, b) is given by
(a) (6, 4) (b) (–3, 3)
(c) (–8, 8) (d) (0, 7)

6. The straight lines $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by - 1 = 0$ are concurrent if the straight line $35x - 22y + 1 = 0$ passes through the point
(a) (a, b) (b) (b, a)
(c) (a, –b) (d) (–a, b)

7. A straight line through the point (2, 2) intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. The equation to the line AB so that the triangle OAB is equilateral is
(a) $x - 2 = 0$ (b) $y - 2 = 0$
(c) $x + y - 4 = 0$ (d) none of these

8. If A(2, 0), B(0, 2) be the ends of the hypotenuse of a right angled isosceles triangle then the third vertex is
(a) (0, 0) or (2, 2) (b) (0, 0) or (1, 1)
(c) (2, 2) or (1, 1) (d) none of these

9. The image of the point (3, 8) in the line $x + 3y = 7$ is
(a) (1, 4) (b) (4, 1)
(c) (–1, –4) (d) (–4, –1)

10. The points (–a, –b), (0, 0), (a, b) and (a^2, ab) are
(a) collinear (b) vertices of a parallelogram
(c) vertices of a triangle (d) none of these

11. The points (0, –1), (–2, 3), (6, 7) and (8, 3) are
(a) collinear
(b) vertices of a parallelogram which is not a rectangle
(c) vertices of a rectangle, which is not a square
(d) none of these

12. The locus of the mid point of the portion intercepted between the axes by the line $x \cos\alpha + y \sin\alpha = p$, where p is a constant is
(a) $x^2 + y^2 = 4p^2$ (b) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$
(c) $x^2 + y^2 = \frac{4}{p^2}$ (d) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{2}{p^2}$

13. If a variable line passes through the point of intersection of the lines $x + 2y - 1 = 0$ and $2x - y - 1 = 0$ and meets the coordinate axes in A and B, then the locus of the mid point of AB is
(a) $x + 3y = 0$ (b) $x + 3y = 10$
(c) $x + 3y = 10xy$ (d) none of these

14. If a line joining two points A(2, 0) and B(3, 1) is rotated about A in anti – clockwise direction through an angle 15°, then the equation of the line in the new position is
(a) $\sqrt{3}x - y = 2\sqrt{3}$ (b) $\sqrt{3}x + y = 2\sqrt{3}$
(c) $x + \sqrt{3}y = 2\sqrt{3}$ (d) none of these

15. If a, b, c are in AP, then ax + by + c = 0 will always pass through a fixed point whose coordinates are

(a) (1, –2) **(b)** (–1, 2)
(c) (1, 2) **(d)** (–1, –2)

16. If a line joining two points A(2, 0) and B(3, 1) is rotated about A in anticlockwise direction through an angle 15° such that the point B goes to C in the new position, then the coordinates of C are

(a) $\left(2+\frac{1}{\sqrt{2}},\ \sqrt{3}\right)$ **(b)** $\left(2+\frac{1}{\sqrt{2}},\sqrt{\frac{3}{2}}\right)$
(c) $\left(2+\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2}\right)$ **(d)** $\left(2-\frac{1}{\sqrt{2}},-\sqrt{\frac{3}{2}}\right)$

17. The distance of the point (3, 5) from the line 2x + 3y – 14 = 0 measured parallel to the line x – 2y = 1 is

(a) $7/\sqrt{5}$ **(b)** $7/\sqrt{13}$
(c) $\sqrt{5}$ **(d)** $\sqrt{13}$

18. In a rhombus ABCD the diagonals AC and BD intersect at the point (3, 4). If the point A is (1, 2) the diagonal BD has the equation

(a) x – y – 1 = 0 **(b)** x + y – 1 = 0
(c) x – y + 1 = 0 **(d)** x + y – 7 = 0

19. A line passes through (2, 2) and is perpendicular to the line 3x + y = 3. Its y – intercept is

(a) 1 / 3 **(b)** 2 / 3
(c) 1 **(d)** 4 / 3

20. The equations of the lines through (–1, –1) and making angle 45° with the line x + y = 0 are given by

(a) $x^2 - xy + x - y = 0$ **(b)** $xy - y^2 + x - y = 0$
(c) xy + x + y = 0 **(d)** xy + x + y + 1 = 0

21. The equations of two sides of a square whose area is 25 square units are 3x – 4y = 0 and 4x + 3y = 0. The equations of the other two sides of the square are

(a) 3x – 4y ± 25 = 0, 4x + 3y ± 25 = 0 **(b)** 3x – 4y ± 5 = 0, 4x + 3y ± 5 = 0
(c) 3x – 4y ± 5 = 0, 4x + 3y ± 25 = 0 **(d)** none of these

22. If the points (1, 3) and (5, 1) are two opposite vertices of a rectangle and the other two vertices lie on the line y = 2x + c, then the value of c is

(a) 4 **(b)** –4
(c) 2 **(d)** none of these

23. The image of the point (1, 3) in the line x + y – 6 = 0 is

(a) (3, 5) **(b)** (5, 3)
(c) (1, –3) **(d)** (–1, 3)

24. A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line x = 3, then their coordinate are

(a) (3, 1), (3, 3) **(b)** (3, 1), (3, 6)
(c) (3, 1), (3, 4) **(d)** none of these

25. The diagonals of a parallelogram PQRS are along the lines x +3y = 4 and 6x –2y =7. Then PQRS must be a

(a) Rectangle **(b)** square
(c) Cyclic quadrilateral **(d)** rhombus

26. The equation of the straight line which bisects the intercepts made by the axes on the lines x + y = 2 and 2x + 3y = 6 is

(a) $2x = 3$ **(b)** $y = 1$
(c) $2y = 3$ **(d)** $x = 1$

27. The foot of the perpendicular on the line $3x + y = \lambda$ drawn from the origin is C. If the line cuts the x – axis and y – axis at A and B respectively then BC: CA is
(a) 1 : 3 **(b)** 3 : 1
(c) 1 : 9 **(d)** 9 : 1

28. The coordinates of two consecutive vertices A and B of a regular hexagon ABCDEF are (1,0) and (2, 0) respectively. The equation of the diagonal CE is
(a) $\sqrt{3}x + y = 4$ **(b)** $x + \sqrt{3}y + 4 = 0$
(c) $x + \sqrt{3}y = 4$ **(d)** none of these

29. ABC is an isosceles triangle in which A is (–1,0) $\angle A = 2\pi/3$, AB = AC and AB is along the x – axis. If $BC = 4\sqrt{3}$ then the equation of the line BC is
(a) $x + \sqrt{3}y = 3$ **(b)** $\sqrt{3}x + y = 3$
(c) $x + y = \sqrt{3}$ **(d)** none of these

30. If a vertex of an equilateral triangle is the origin and the side opposite to it has the equation $x + y = 1$ then the orthocentre of the triangle is
(a) $\left(\frac{1}{3}, \frac{1}{3}\right)$ **(b)** $\left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right)$
(c) $\left(\frac{2}{3}, \frac{2}{3}\right)$ **(d)** none of these

Answers

Practice Assignment – I

1. (i) 2
(ii) –3
(iii) $-\frac{1}{6}$
(iv) $\frac{1}{2}$

2. (i) $\tan^{-1}\left(\frac{11}{23}\right)$ (ii) $-\sqrt{3}$ (iii) 135° (iv) 1 and 2, or $\frac{1}{2}$ and 1, –1 and –2, or $-\frac{1}{2}$ and–1

3. (i) $y = 5$
(ii) $x + 3 = 0$

4. (i) $x - 2y = 0$
(ii) $5x - 2y + 7 = 0$

5. (i) $2x + 3y = 0$
(ii) $3x - 2y - 6 = 0$
(iii) $x + y - 3 = 0$
(iv) $7x - 3y + 25 = 0$

(v) $x + y + 3 = 0$

6. (i) $x - y + 2 = 0$

7. $x - \sqrt{3}\,y + 3 + 2\sqrt{3} = 0$

8. (i) $x - y + 3 = 0$

(ii) $x - y = 2$

9. $x - 2y - 14 = 0$ or $x + 6y - 6 = 0$

10. $\frac{x}{6} + \frac{y}{10} = 1$

11. $\frac{1}{2}$, 2, -4, $45°$

12. $4x + y - 11 = 0$, $x - 4y + 3 = 0$

13. $2x - 3y + 14 = 0$

14. -2, (1,2)

15. $3, -\frac{3}{2}$

16. $27x + 8y = 180$

17. $x + y = 0$

18. $8x + 3y - 10 = 0$ and $8x - 3y + 2 = 0$

19. $y = 4x + 6000$

20. $y = \frac{4}{5}x + 570$; Rs 890

21. $3x - 4y + 8 = 0$

22. $5x - y + 20 = 0$

23. $(1 + n)x + 3(1 + n)y = n + 11$

24. $x + y = 5$

25. $x + 2y - 6 = 0$, $2x + y - 6 = 0$

26. (i) $y = -\frac{1}{7}x + 0, -\frac{1}{7}, 0$; (ii) $y = -2x + \frac{5}{3}, -2, \frac{5}{3}$; (iii) $y = 0x + 0, 0, 0$

27. (i) $\frac{x}{4} + \frac{y}{6} = 1, 4, 6$; (ii) $\frac{x}{\frac{3}{2}} + \frac{y}{-2} = 1, \frac{3}{2}, -2$

(iii) $y = -\frac{2}{3}$, intercept with y-axis $= -\frac{2}{3}$ and no intercept with x-axis

28. $30°$ and $150°$

29. $(\sqrt{3} + 2)x + (2\sqrt{3} - 1)y = 8\sqrt{3} + 1$ or $(\sqrt{3} - 2)x + (1 + 2\sqrt{3})y = -1 + 8\sqrt{3}$

30. $\left(\frac{13}{5}, 0\right)$

Practice Assignment- II

1. (i) $x \cos 120° + y \sin 120° = 4$, 4, $120°$

(ii) $x \cos 90° + y \sin 90° = 2$, 2, $90°$

(iii) $x \cos 315° + y \sin 315° = 2\sqrt{2}$, $2\sqrt{2}$, $315°$

2. **(i)** **$(\sqrt{3}-1)x + (\sqrt{3}+1)y - 12 = 0$**

(ii) **$x + \sqrt{3}y = 24$**

3. **$\left(\frac{8}{\sqrt{3}}, 0\right)$, (0, 8), 4, 30°**

4. **(i) $2x + 3y - 1 = 0$ (ii) $4x - 3y - 11 = 0$**

5. **$3x - 4y - 1 = 0$, (7,5), (–1, –1)**

6. **$\frac{132}{12\sqrt{3}+5}$**

8. **(i) 15° or 75° (ii) 0°**

9. **$\frac{23\sqrt{5}}{18}$ units**

10. **(i)** **k = 10 ; (–1, 2) (ii) 5**

13. **$\left(1, \frac{12}{5}\right), \left(-3, \frac{16}{5}\right)$**

14. **(3, 1), (–7, 11)**

16. **(i)** **$\frac{33}{10}$** **(ii) $\frac{65}{17}$ units** **(iii) $\frac{1}{\sqrt{2}}\left|\frac{p+r}{\ell}\right|$ units**

17. **$2x - y + 6 = 0$ or $2x - y - 14 = 0$**

20. $\left(\frac{-1}{14}, \frac{73}{28}\right), \left(\frac{1}{16}, \frac{77}{32}\right)$

21. 49 sq. units

23. $18x + 12y + 11 = 0$

24. $3x - 4y = 0$, $3x - 4y - 10 = 0$

26. $119x + 102y = 125$

Practice Assignment -III

1. opposite
2. $|a| < 1$
3. –
4. $\left(\frac{-4\sqrt{2}}{3}, \frac{5\sqrt{2}}{6}\right)$
5. (i) $y + 2 = 0$, $\sqrt{3}x - y = 2 + 3\sqrt{3}$ (ii) $3x - y = 7$, $x + 3y = 9$
6. $y - 3 = (2 - \sqrt{3})(x - 2)$, $y - 3 = (2 + \sqrt{3})(x - 2)$
7. $7x + y = 9$ or $x - 7y + 13 = 0$
8. $9x - 7y = 1$, $7x + 9y = 73$
9. $x + y - 2 = 0$ and $x = y$
10. (3, 4)
11. $43x - 29y = 71$
12. $23x + 23y = 11$

13. $4x + 5y = 9$
14. $x + y = 3$
15. (a) $y = 3$ (b) $x = 4$ (c) $3x + 4y = 24$
16. $4x - 3y + 2 = 0$
17. $x - y = 5$
18. (i) $(\sqrt{5} - 2\sqrt{2})x + (\sqrt{5} + \sqrt{2})y = 3\sqrt{5} - 2\sqrt{2}$
    (ii) Same as (i)
19. $11x - 3y + 9 = 0$
20. $3x = 19$
22. $\left(\frac{-1}{10}, \frac{37}{10}\right)$
23. $\left(\frac{-46}{5}, \frac{-8}{5}\right)$
24. $9x + 12y = 14$
25. $(-2, 9)$

Objective Assignment

1	C	11	C	21	A
2	A	12	B	22	B
3	C	13	C	23	A
4	C	14	A	24	B
5	C	15	A	25	D
6	A	16	B	26	B
7	B	17	C	27	D
8	A	18	D	28	C
9	C	19	D	29	A
10	A	20	D	30	A